CHAPTER II

RESOLVENTS AND SPECTRAL FUNCTIONS OF LARGE SAMPLE COVARIANCE MATRICES

Consider a sample $\mathfrak{X} = \{\mathbf{x}_m\}$ of size N from a population with the covariance matrix $\Sigma = \text{cov}\,(\mathbf{x}, \mathbf{x})$, random Gram matrix S and sample covariance matrix C of the form

$$S = N^{-1}\sum_{m=1}^{N} \mathbf{x}_m \mathbf{x}_m^T, \quad \text{and} \quad C = N^{-1}\sum_{m=1}^{N} (\mathbf{x}_m - \bar{\mathbf{x}})(\mathbf{x}_m - \bar{\mathbf{x}})^T,$$

where $\bar{\mathbf{x}}$ is the sample mean vector. We investigate the resolvents

$$H_0 = H_0(z) = (I - zS)^{-1} \quad \text{and} \quad H = H(z) = (I - zC)^{-1}$$

of large matrices S and C. These resolvents are supposed to be functions of a complex parameter z. Analytical properties of these are of interest as a tool for obtaining the empirical distribution function of their eigenvalues. For example, let A be any real symmetric positive definite matrix of size $n \times n$ and $h(z) = (I - zA)^{-1}$. Then the empirical distribution function of eigenvalues λ_i of A

$$F(u) = \sum_{i=1}^{n} \text{ind}(\lambda_i \le u),$$

can be calculated as follows:

$$F(u) = \frac{1}{\pi} \lim_{\varepsilon \to +0} \ \text{Im} \int_0^u h(z^{-1}) z^{-1} dv,$$

where $z = v - i\varepsilon$.

We assume that the population is such that all components of the observation vector $\mathbf{x}$ have the fourth moments and $\mathbf{E}\mathbf{x} = 0$. To

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measure the remainder terms of our asymptotic approach, we define the maximum fourth moment of a projection of $\mathbf{x}$ onto non-random axes (defined by vectors $\mathbf{e}$ of unit length)

$$M = \sup_{|\mathbf{e}|=1} \mathbf{E}\ (\mathbf{e}^T\mathbf{x})^4 > 0 \tag{1}$$

and special measures of the quadratic from variance

$$\nu = \sup_{\|\Omega\|=1} \text{var}\ (\mathbf{x}^T\Omega\mathbf{x}/n), \quad \gamma = \nu/M, \tag{2}$$

where Ω are non-random symmetric positive semidefinite matrices of unit spectral norm.

Consider the region of the complex plane

$$\mathfrak{G} = \{z : \text{Re}\ z < 0 \ \text{ or } \ \text{Im}\ z \neq 0\}.$$

Denote

$$\alpha = \alpha(z) = \begin{cases} 1 & \text{if Re } z \leq 0, \\ |z|/|\text{Im } z| & \text{if Re } z > 0 \text{ and Im } z \neq 0. \end{cases}$$

To estimate expressions involving the resolvent we will use the following inequalities.

Remark 1. Let A be a real symmetric matrix, $\mathbf{q}$ be a vector with n complex components, and $\mathbf{q}^H$ be the Hermitian conjugate vector (here and in the following, the superscript H denotes the Hermitian conjugate). If $u \geq 0$ and $z \in \mathfrak{G}$, then

$$|1 - zu|^{-1} \leq \alpha, \quad \|(I - zA)^{-1}\| \leq \alpha,$$
$$\|I - (I - zA)^{-1}\| \ \leq \alpha,$$
$$|1 - \mathbf{q}^H(I - zA)^{-1}\mathbf{q}|^{-1} \leq \alpha.$$

For example, if $z \in \mathfrak{G}$, then $\|H_0(z)\| \leq \alpha$ and $\|H(z)\| \leq \alpha$.

Spectral Functions of Random Gram Matrices

Our main tool is the method of alternating elimination of independent sample vectors. In this chapter we assume that the sample size

$N > 1$. Let $\mathbf{x}_m$ be one of the sample vectors and $\mathbf{e}$ be a non-random complex vector of length 1, $\mathbf{e}^H\mathbf{e} = 1$.

Denote

$$\begin{gathered} h_0(z) = \mathbf{E}\ n^{-1}\mathrm{tr}\ H_0(z), \quad y = n/N, \quad s_0(z) = 1 - y + yh_0(z), \\ S^m = S - N^{-1}\mathbf{x}_m\mathbf{x}_m^T, \quad H_0^m = (I - zS^m)^{-1}, \\ \varphi_m = \varphi_m(z) = \mathbf{x}_m^T H_0^m \mathbf{x}_m/N, \quad \psi_m = \psi_m(z) = \mathbf{x}_m^T H_0 \mathbf{x}_m/N, \\ v_m = \mathbf{e}^H H_0 \mathbf{x}_m, \quad u_m = \mathbf{e}^H H_0 \mathbf{x}_m, \quad m = 1, \ldots, N. \end{gathered} \tag{3}$$

Remark 2. If $z \in \mathfrak{G}$, the following relations are valid

$$\begin{gathered} H_0 = H_0^m + zH_0^m\mathbf{x}_m\ \mathbf{x}_m^T H_0/N, \quad H_0\mathbf{x}_m = (1 - z\psi_m)H_0^m\mathbf{x}_m, \\ u_m = v_m + z\psi_m v_m = v_m + z\varphi_m u_m, \\ \psi_m = \varphi_m + z\varphi_m\psi_m, \qquad (1 + z\psi_m)(1 - z\varphi_m) = 1, \\ |1 - z\varphi_m|^{-1} \le \alpha, \quad |u_m| \le \alpha|v_m|, \quad m = 1, \ldots, N. \end{gathered} \tag{4}$$

Lemma 2.1. *If $z \in \mathfrak{G}$, then*

$$\begin{gathered} \mathbf{E}\ v_m = 0, \quad \mathbf{E}\ v_m^4 \le M\alpha^4, \quad 1 + z\mathbf{E}\ \psi_m = s_0(z), \\ |\mathbf{E}\ u_m|^2 \le \sqrt{M}|z|^2\alpha^2\ \mathrm{var}\ \psi_m, \quad m = 1, \ldots, N. \end{gathered} \tag{5}$$

Proof. The first two inequalities immediately follow from the independence of $\mathbf{x}_m$ and H_0^m and (1). Now, we have

$$\begin{aligned} z\mathbf{E}\ \psi_m &= \mathbf{E}\ z\mathbf{x}_m^T H_0\mathbf{x}_m/N = \mathbf{E}\ z\ \mathrm{tr}\ (SH_0)/N = \mathbf{E}\ (H_0 - I)/N \\ &= y(h_0(z) - 1) = s_0(z) - 1. \end{aligned}$$

We notice that $\mathbf{E}\, u_m = \mathbf{E}\, v_m\Delta_m$, where $\Delta_m = \psi_m - \mathbf{E}\,\psi_m$. Applying the Schwarz inequality we obtain the last lemma statement. □

Define the variance of a complex variable

$$\mathrm{Var}\ (z) = \mathbf{E}\ (z - \mathbf{E}\ z)(z^* - \mathbf{E}\ z^*),$$

and denote

$$\tau = \sqrt{M}|z|, \quad \delta = 2\alpha^2\ y^2(\gamma + \tau^2\alpha^4/N).$$

To estimate variances of functionals uniformly depending on a large number of independent variables we use the technique of expanding with respect to martingale differences. It will be sufficient to cite the Burkholder inequality (see in [57]). However, we present the following statement with the full proof.

Lemma 2.2. *Given a set $\mathfrak{X} = \{X_1, \dots, X_N\}$ of independent variables, we consider a function $\varphi(\mathfrak{X})$ such that $\varphi(\mathfrak{X}) = \varphi^m(\mathfrak{X}) + \Delta_m(\mathfrak{X})$, where $\varphi^m(\mathfrak{X})$ does not depend on X_m, $m = 1, \dots, N$. If second moments exist for $\varphi(\mathfrak{X})$ and $\Delta_m = \Delta_m(\mathfrak{X})$, $m = 1, \dots, N$, then*

$$\text{var } \varphi(\mathfrak{X}) \leq \sum_{m=1}^{N} \mathbf{E} \, (\Delta_m - \mathbf{E}_m \Delta_m)^2,$$

where $\mathbf{E}_m$ stands for the expectation calculated by integration with respect to the distribution of the variable X_m only.

Proof. Denote by F_m the distribution function of X_m, $m = 1, \dots, N$, and let $dF^{(m)}$ denote the product $dF_1 dF_2 \dots dF_m$, $m = 1, \dots, N$. Consider the martingale differences

$$\beta_1 = \varphi - \int \varphi dF_1, \quad \beta_m = \int \varphi dF^{m-1} - \int \varphi dF^m, \quad m = 2, \dots, N,$$

where $\varphi = \varphi(\mathfrak{X})$. In view of the independence of $X_1, \dots, X_N$, it can be readily seen that $\mathbf{E} \, \beta_i \beta_j = 0$ if $i \neq j$, $i, j = 1, \dots, N$. Majorizing the square of the first moment by the second moment, we obtain

$$\begin{aligned} \mathbf{E} \, \beta_m^2 &= \mathbf{E} \, [\int (\varphi - \int \varphi dF_m) dF^{m-1}]^2 \\ &\leq \mathbf{E} \, \int (\varphi - \int \varphi dF_m)^2 dF^{m-1} = \mathbf{E} \, (\varphi - \mathbf{E}_m \varphi)^2. \end{aligned}$$

We have $\mathbf{E} \, \beta_m^2 \leq \mathbf{E} \, (\Delta_m - \int \Delta_m dF_m)^2$. The statement of the lemma follows. □

Lemma 2.3. *If $z \in \mathfrak{G}$, then we have* $\text{Var } (\mathbf{e}^H H_0 \mathbf{e}) \leq \tau^2 \alpha^6 / N$, $\text{Var } \varphi_m \leq M\delta/2$, *and* $\text{Var } \psi_m \leq aM\alpha^4 \delta$, $m = 1, \dots, N$, *where a is a numerical constant.*

Proof. From (4) it follows that $\mathbf{e}^H H_0 \mathbf{e} = \mathbf{e}^H H_0^m \mathbf{e} + z v_m u_m / N$, $m = 1, \dots, N$. Using Remark 2, (5) and (1), we find that

$$\begin{aligned} \text{Var } (\mathbf{e} H_0 \mathbf{e}) &\leq |z|^2 \sum_{m=1}^{N} \mathbf{E} \, |v_m u_m|^2 / N^2 \\ &\leq |z|^2 \alpha^2 \mathbf{E} \, |v_m|^4 / N \leq \tau^2 \alpha^6 / N. \end{aligned}$$

Now we fix an integer m, $m = 1, \dots, N$. Denote $\Omega = \mathbf{E}\ H_0^m$, $\Delta H_0^m = H_0^m - \Omega$. Since Ω is non-random we have

$$\text{Var}\ \varphi_m = \mathbf{E}\ |\mathbf{x}_m^T \Delta H_0^m \mathbf{x}_m|^2/N^2 + \text{Var}\ (\mathbf{x}_m^T \Omega \mathbf{x}_m)/N^2. \tag{6}$$

Notice that H_0^m is a matrix of the form H_0 with N less by 1 if we replace the argument t by $t' = (1-N^{-1})t$. We apply the first statement of this lemma to estimate the conditional variance Var $(\tilde{\mathbf{e}}^H H_0^m \tilde{\mathbf{e}})$ under fixed $\mathbf{x}_m$, where $\tilde{\mathbf{e}}$ is a unit vector directed along $\mathbf{x}_m$, and find that the first summand in (6) is not greater than

$$\mathbf{E}\ |\mathbf{x}_m|^4 \tau^2 \alpha^6/N^3 \le M\tau^2\alpha^6 y^2/N.$$

To estimate the second summand, we introduce the parameter (2) to (6) and obtain that the second summand in (6) is not greater than $\|\Omega\|^2 M n^2 \gamma/N^2 \le M\alpha^2 y^2 \gamma$. Summing both summands, we obtain the right hand side of the second inequality in the statement of the lemma.

Further, the equation connecting φ_m and ψ_m in Remark 2 can be rewritten in the form

$$(1 - z\varphi_m)\Delta\psi_m = (1 + z\mathbf{E}\ \psi_m)\Delta\varphi_m - z\mathbf{E}\ \Delta\varphi_m \Delta\psi_m,$$

where $\Delta\varphi_m = \varphi_m - \mathbf{E}\ \varphi_m$ and $\Delta\psi_m = \psi_m - \mathbf{E}\ \psi_m$. We square the absolute values of both parts of this equation and take into account that $|1 - z\varphi_m|^{-1} \le \alpha$, and $|1 + z\mathbf{E}\ \psi_m| \le \alpha$. It follows that

$$\text{Var}\ \psi_m \le \alpha^4 \text{Var}\ \varphi_m + \alpha^2 |z|^2 \text{Var}\ \varphi_m \text{Var}\ \psi_m.$$

Here in the second summand of the right hand side,

$$|z|\ \text{Var}\ \psi_m \le \mathbf{E}\ |z\psi_m|^2 = \mathbf{E}\ |z\varphi_m(1 - z\varphi_m)^{-1}|^2 \le (1+\alpha)^2.$$

But $\alpha \ge 1$. It follows that Var $\psi_m \le 5\alpha^4$Var φ_m. The last statement of our lemma is proved. □

Remark 3. If $z \in \mathfrak{G}$ and $u \ge 0$, we have $|1 - zs_0(z)u|^{-1} \le \alpha$.

Indeed, first let Re $z \le 0$. We single out a sample vector $\mathbf{x}_m$. Using (4), we obtain

$$s_0(z) = 1 + z\mathbf{E}\ \psi_m = \mathbf{E}\ (1 - z\varphi_m)^{-1},$$
$$zs_0(z) = \mathbf{E}\ (z^{-1} - \varphi_m^{-1})^{-1} = \mathbf{E}\ r(z^{-1} - \varphi_m^{-1})^*,$$

where $r \geq 0$. We examine that Re $\varphi_m \geq 0$ for Re $z \leq 0$. It follows that in this case, Re $zs_0(z) \leq 0$ and $|1 - zs_0(z)u| \geq 1$. Now, let Re $z > 0$ and Im $z \neq 0$. The sign of Im z coincides with the sign of Im $h_0(z)$ and with the sign of Im $s_0(z)$. Therefore,

$$|1 - zs_0(z)u| \geq |z| \ \left|\text{Im } z/|z|^2 + u \text{ Im } s_0(z)\right| \geq |\text{Im } z/z| = \alpha^{-1}.$$

Our remark is grounded for the both cases. □

THEOREM 2.1. *For any population in which the four moments of all variables exist, for any $z \in \mathfrak{G}$, we have*

$$\text{Var } (\mathbf{e}^H H_0(z)\mathbf{e}) \leq \tau^2\alpha^6/N,$$
$$\mathbf{E} \ H_0(z) = (I - zs_0(z)\Sigma)^{-1} + \Omega_0,$$

where $\|\Omega_0\| \leq o_N \stackrel{\text{def}}{=} a\tau^2\alpha^4(\sqrt{\delta} + \alpha/N)$, and a is a numerical constant.

Proof. The first statement of the theorem is proved in Lemma 2.3. To prove the second one, we fix an integer m, $m = 1, \ldots, N$, and multiply both sides of the second relation in (4) by $\mathbf{x}_m\mathbf{x}_m^T$. It follows

$$H_0\mathbf{x}_m\mathbf{x}_m^T = H_0^m\mathbf{x}_m\mathbf{x}_m^T + zH_0^m\mathbf{x}_m\mathbf{x}_m^T H_0\mathbf{x}_m\mathbf{x}_m^T/N.$$

Multiplying by z, we calculate the expectation values

$$\begin{aligned} z\mathbf{E} \ H_0\mathbf{x}_m\mathbf{x}_m^T &= \mathbf{E} \ zH_0S = \mathbf{E} \ (H_0 - I) \\ &= z\mathbf{E} \ H_0^m\Sigma + z\mathbf{E} \ H_0^m\mathbf{x}_m\mathbf{x}_m^T(1 - z\psi_m). \end{aligned}$$

Substituting $z\psi_m = s_0(z) - 1 + z\Delta\psi_m$, where $\Delta\psi_m = \psi_m - \mathbf{E} \ \psi_m$, we obtain

$$\mathbf{E} \ H_0 = I + zs_0(z)\mathbf{E} \ H_0^m\Sigma + \Omega_1,$$

where $\Omega_1 = z^2\mathbf{E} \ H_0^m\mathbf{x}_m\mathbf{x}_m^T\Delta\psi_m$. Using (4) once more to replace H_0^m, we find that

$$(I - zs_0(z)\Sigma)\mathbf{E} \ H_0 = I + \Omega_1 + \Omega_2,$$

where $\Omega_2 = zs_0(z)\mathbf{E}\ (H_0^m - H_0)\Sigma$. Denote $R = (I - zs_0(z)\Sigma)^{-1}$. By Remark 3, $\|R\| \le \alpha$. Multiplying by R we obtain $\mathbf{E}\ H_0 = R + \Omega$, where $\Omega = R\Omega_1 + R\Omega_2$. We notice that Ω is a symmetric matrix and, consequently, its spectral norm equals $|\mathbf{e}^H\Omega\mathbf{e}|$, where $\mathbf{e}$ is one of its eigenvalues. Denote $\mathbf{f} = R\ \mathbf{e}$, $\mathbf{f}^H\mathbf{f} \le \alpha$. We have $\|\Omega\| = |\mathbf{f}^T\Omega_1\mathbf{e} + \mathbf{f}^T\Omega_2\mathbf{e}|$. Now,

$$|\mathbf{f}^H\Omega_1\mathbf{e}| = |z|\ \mathbf{E}\ |\mathbf{f}^H H_0^m\mathbf{x}_m(\mathbf{x}_m^T\mathbf{e})\Delta\psi_m| \le |z|^2 \left(\mathbf{E}\ |\mathbf{f}^H H_0^m\mathbf{x}_m|^4\ \mathbf{E}\ |\mathbf{x}_m^T\mathbf{e}|^4\right)^{1/4} \sqrt{\mathrm{Var}\ \psi_m}. \quad (7)$$

Here $\mathbf{E}\ |\mathbf{f}^H H_0^m\mathbf{x}_m|^4 \le M\mathbf{E}\ |\mathbf{f}^H H_0^m\ H_0^{m*}\mathbf{f}|^2 \le M\alpha^8$, $\mathbf{E}\ |\mathbf{x}_m^T\mathbf{e}|^4 \le M$, Var $\psi_m \le aM\alpha^4\sqrt{\delta}$. It follows that the left hand side of (7) is not greater than $M|z|^2\alpha^4\sqrt{\delta}$. Then

$$\|\Omega_2\| \le |\mathbf{f}^H\Omega_2\mathbf{e}| = |z|^2\ |s_0(z)|\ |\mathbf{E}\ \mathbf{f}^H H_0\mathbf{x}_m(\mathbf{x}_m^T H_0^m\Sigma\mathbf{e})/N| \le |z|^2\ |s_0(z)| \left(\mathbf{E}\ |\mathbf{f}^H H_0\mathbf{x}_m|^2\ \mathbf{E}\ |\mathbf{x}_m^T H_0^m\Sigma\mathbf{e}|^2\right)^{1/2}/N.$$

Here by (4), we have that $|s_0(z)| = |\mathbf{E}\ (1 - z\varphi)^{-1}| \le \alpha$; using the second of inequalities (4), we find that

$$\mathbf{E}\ |\mathbf{f}^H H_0\mathbf{x}_m|^2 \le |\mathbf{f}|^2\ \alpha^2\ \mathbf{E}\ |\mathbf{e}_1^H H_0^m\mathbf{x}_m|^2 \le \sqrt{M}\alpha^6,$$

where $\mathbf{e}_1 = \mathbf{f}/|\mathbf{f}|$. Obviously,

$$\mathbf{E}\ \ |\mathbf{x}_m^T H_0^m\Sigma\mathbf{e}|^2 = \sqrt{M}(\mathbf{e}^T\Sigma H_0^m\Sigma H_0^{m*}\Sigma\mathbf{e}) \le M^{3/2}\alpha^2.$$

Therefore $|\mathbf{f}^T\Omega_2\mathbf{e}|$ is not greater than $M|z|^2\alpha^5/N$. We obtain the required upper estimate of $\|\Omega\|$. This completes the proof. □

Corollary. For $z \in \mathfrak{G}$ we have

$$h_0(z) = n^{-1}\mathrm{tr}\ (I - zs(z)\Sigma)^{-1} + \omega, \quad (8)$$

where $|\omega| \le o_N$.

Restricted dependence condition.

Note that the boundedness of the moments M essentially restricts the dependence of variables. Indeed, let Σ be a correlation matrix with the Bayes distribution of the correlation coefficients that is uniform on the segment $[-1,1]$. Then the Bayes mean $\mathbf{E}\ M \geq \mathbf{E}\ n^{-1}\mathrm{tr}\ \Sigma^2 \geq (n+2)/3$. In the case of $N(0,\Sigma)$ with the matrix Σ all of whose entries are 1, the value $M = 3n^2$.

Let us prove that relation (8) can be established with accuracy to terms, in which the moments of the variables are restricted only in a set.

Denote

$$\Lambda_k = n^{-1}\mathrm{tr}\ \Sigma^k, \quad Q_k = \mathbf{E}\ (\mathbf{x}^2/n)^k, \quad W = n^{-2} \sup_{\|\Omega\|=1} \mathbf{E}\ (\mathbf{x}^T \Omega \mathbf{x}')^4,$$

$k \geq 0$, where $\mathbf{x}$ and $\mathbf{x}'$ are independent vectors and Ω are non-random symmetric positive semidefinite matrices of unit spectral norm.

Remark 4. If $t \geq 0$, then relation (8) holds with the remainder term ω such that $\omega^2/2 \leq \left[Q_2 y^2(\nu + Wt^2/N) + W/N^2\right] t^4$.

Example. Let $\mathbf{x} \sim \mathbf{N}(0,\Sigma)$. Denote $\Lambda_k = n^{-1}\mathrm{tr}\ \Sigma^k$, $k = 1, 2, \ldots$ For normal $\mathbf{x}$, we have

$$M = 3\|\Sigma\|^2, \quad Q_2 = \Lambda_1^2 + 2\Lambda_2/n,$$

$W = 3(\Lambda_2^2 + 2\Lambda_4/n)$, $\nu = 2\Lambda_2/n$. Assume that $\Sigma = I + \rho E$, where E is a matrix all of whose entries are 1, and $0 \leq \rho \leq 1$. Then $M = 3(1+n\rho)^2$, $\Lambda_1 = 1+\rho$, $\Lambda_2 = 1+2\rho+n\rho^2$, $\Lambda_k \leq a_k + b_k\rho^k n^{k-1}$, where a_k and b_k are positive numbers independent of n, and all $Q_k < c$, where c does not depend on n. If $\rho = \rho(n) = n^{-3/4}$ as $n \to \infty$, then $M \to \infty$, whereas the values Λ_3, Λ_4 and Q_3 remain finite. Nevertheless, the quantities $\nu = O(n^{-1})$, and $\omega \to 0$.

Spectral Functions of Sample Covariance Matrices

To pass to matrices C we use the relation $C = S - \bar{\mathbf{x}}\,\bar{\mathbf{x}}^T$ and show that this difference does not influence the leading parts of spectral equations and only effects the remainder terms.

Denote

$$H = H(z) = (I - zC)^{-1}, \quad h(z) = n^{-1}\mathrm{tr}\ H(z),$$
$$s(z) = 1 - y + yh(z), \quad V = \mathbf{e}^H H(z)\bar{\mathbf{x}}, \quad U = \mathbf{e}^H H(z)\bar{\mathbf{x}},$$
$$\Phi = \bar{\mathbf{x}}^T H(z)\bar{\mathbf{x}}, \quad \Psi = \bar{\mathbf{x}}^T H(z)\bar{\mathbf{x}}, \tag{9}$$

where $\mathbf{e}$ is a complex vector with $\mathbf{e}^H\mathbf{e} = 1$ where the superscript H stands for the Hermitian conjugation.

Remark 5. If $z \in \mathfrak{G}$, then

$$H = H_0 - zH_0\bar{\mathbf{x}}\bar{\mathbf{x}}H, \quad U = V - z\Phi U = V - z\Psi V,$$
$$(1 + z\Phi)(1 - z\Psi) = 1, \quad |1 - z\Psi| \le \alpha.$$

Indeed, the first three identities can be checked straightforwardly. The fourth statement follows from Remark 1.

Remark 6. If $z \in \mathfrak{G}$, then

$$q \stackrel{\text{def}}{=} |z| \ \bar{\mathbf{x}}^T H_0 H_0^* \bar{\mathbf{x}} \le \alpha^2.$$

Let us derive this inequality. Let $\mathbf{w}$ be a complex vector. We denote complex scalars $\mathbf{w}^T\mathbf{w}$ by $\mathbf{w}^2$, and the real product $\mathbf{w}^H\mathbf{w}$ by $|\mathbf{w}|^2$. Denote the matrix product $Z^H Z$ by $|Z|^2$. Let, by definition, $\Omega^2 = I - zC, \ \ \bar{\mathbf{y}} = \Omega^{-1}\bar{\mathbf{x}}, \ \ \mathbf{a} = |H|\bar{\mathbf{x}}$. Then

$$H_0 = (I - zC - z\ \bar{\mathbf{x}}\ \bar{\mathbf{x}}^T)^{-1} = \Omega^{-1}(I - z\ \bar{\mathbf{y}}\bar{\mathbf{y}}^H)^{-1}\Omega^{-1},$$
$$\bar{\mathbf{y}}^H\bar{\mathbf{y}} = \bar{\mathbf{x}}^T(I - zC)^{-1}\bar{\mathbf{x}} = \mathbf{a}^T(I - zC^*)\mathbf{a}.$$

The original expression

$$q = |z|\bar{\mathbf{x}}^T H_0 H_0^* \bar{\mathbf{x}} = |z| \cdot |\Omega^{-1}(I - z\bar{\mathbf{y}}\bar{\mathbf{y}}^H)^{-1}\bar{\mathbf{y}}|^2$$
$$= |z| \cdot |\bar{\mathbf{y}}^H\Omega^{-2}\bar{\mathbf{y}}|^{-2}|1 - z\bar{\mathbf{y}}^2|^{-2} = |z|\mathbf{a}^2 \ \left|1 - z\mathbf{a}^2 + |z|^2\mathbf{a}^T C\mathbf{a}\right|^{-2}.$$

Denote $t = \mathbf{a}^2/(1 + |z|^2\mathbf{a}^T C\mathbf{a})$. Let $z \neq 0$. If Re $z \le o$, then $q \le |z|\mathbf{a}^2 \le 1$. If Re $z \ge 0$ then the quantity $q \le |z|t^2/|1 - zt|^2$. The maximum of the right hand side of this inequality is attained for $t = 1/\text{Re } z$ and it equals $q = q_{\max} = |z|^2|\text{Im } z|^{-2} = \alpha^2(z)$. This is our assertion. □

LEMMA 2.4. *If* $z \in \mathfrak{G}$ *then* $|z|$ var $V \le 2\tau\alpha^4(1 + \tau\alpha^2)/N$.

Proof. To use Lemma 2.2, we single out one of sample vectors, say, $\mathbf{x}_m$. Denote $\tilde{\mathbf{x}} = \bar{\mathbf{x}} - N^{-1}\mathbf{x}_m$. We have

$$V = \mathbf{e}^H H_0 \bar{\mathbf{x}} = \mathbf{e}^H H_0^m \tilde{\mathbf{x}} + \mathbf{e}^H H_0 \mathbf{x}_m/N + z\mathbf{e}^H H_0^m \mathbf{x}_m \mathbf{x}_m^T H_0 \tilde{\mathbf{x}}/N,$$

where the first summand does not depend on $\mathbf{x}_m$. Denote $w_m = \mathbf{x}_m^T H_0^m \tilde{\mathbf{x}}$. By Lemma 2.2, we have

$$|z| \text{ var } V \leq |z| \ N^{-2} \sum_{m=1}^{N} \mathbf{E} \ |u_m(1 + zw_m)|^2.$$

But

$$\mathbf{E} \ |u_m|^2 \leq \sqrt{M}\alpha^4, \quad \mathbf{E} \ |zu_m w_m|^2 \leq \sqrt{M}\alpha^4 |z|^2 (\mathbf{E} \ |w_m|^4)^{1/2}.$$

The expression

$$|z|^2 \mathbf{E} \ |w_m|^4 \leq M|z|^2 \mathbf{E} \ (\tilde{\mathbf{x}}^T H_0^m H_0^{m*} \tilde{\mathbf{x}})^2 \leq Mq',$$

where q' can be reduced to the form of the expression for q in Remark 6 with the number N less by unit and the argument $z' = (1 - N^{-1})z$. From Remark 6, it follows that $q' \leq \alpha^2$ and, consequently, $|z|^2 \mathbf{E} \ |w_m|^4 \leq M\alpha^4$. We obtain the required upper estimate of $|z|$ var V. The lemma is proved. □

Lemma 2.5. *If $z \in \mathfrak{G}$, then $\|\mathbf{E} \ H - \mathbf{E} \ H_0\| \leq a \ \omega$, where $\omega^2 = \tau^2 \alpha^6 y(\tau^2 \alpha^2 \delta + (1 + \tau\alpha^2)/N)$, and a is a numerical coefficient.*

Proof. In view of the symmetry of matrices H and H_0, we have

$$\begin{aligned} \|\mathbf{E} \ H - \mathbf{E} \ H_0\| &\leq \|z\mathbf{E} \ H\bar{\mathbf{x}}\bar{\mathbf{x}}^T H_0\| = \mathbf{E} \ |zVU| \\ &\leq \left(\mathbf{E} \ |zV^2| \ \mathbf{E} \ |zU^2|\right)^{1/2} \end{aligned}$$

and

$$\begin{aligned} \mathbf{E} \ |zV^2| &= |z \ \mathbf{E} \ V^2| + |z| \text{ Var } V, \\ \mathbf{E} \ |zU^2| &= |z| \ \alpha^2 \mathbf{E} \ \bar{\mathbf{x}}^2 \leq \sqrt{M}|z|\alpha^2 y. \end{aligned}$$

But for any $m = 1, \ldots, N$, $\mathbf{E} \ V = \mathbf{E} \ \mathbf{e}^T H_0 \bar{\mathbf{x}} = |\mathbf{E} \ u_m|$. From (5) and Lemma 2.3 it follows that $|\mathbf{E} \ u_m|^2 \leq aM^{3/2}|z|^2\alpha^6\delta$, where a is a number. Gathering up these estimates we obtain the statement of Lemma 2.5. □

LEMMA 2.6. *If $z \in \mathfrak{G}$ and $u > 0$, then* $|1 - zs(z)u|^{-1} \leq \alpha$.

Proof. Denote

$$C' = C - N^{-1}(\mathbf{x}_m - \bar{\mathbf{x}})(\mathbf{x}_m - \bar{\mathbf{x}})^T, \quad H' = (I - zC')^{-1},$$
$$\Phi' = (\mathbf{x}_m - \bar{\mathbf{x}})^T H'(\mathbf{x}_m - \bar{\mathbf{x}}_m)^T/N,$$
$$\Psi' = (\mathbf{x}_m - \bar{\mathbf{x}})^T H(\mathbf{x}_m - \bar{\mathbf{x}})/N.$$

We examine the identities

$$H = H' + zH(\mathbf{x}_m - \bar{\mathbf{x}})(\mathbf{x}_m - \bar{\mathbf{x}})^T H'/N, \quad (1 + z\Phi')(1 - z\Psi') = 1.$$

It follows

$$\begin{aligned} s(z) &= 1 + y(h(z) - 1) = 1 + \mathbf{E}\ N^{-1}\text{tr }(H - I) \\ &= 1 + z\mathbf{E}\ N^{-1}\text{tr } HC = 1 + \mathbf{E}\ N^{-1}(\mathbf{x}_m - \bar{\mathbf{x}})^T H(\mathbf{x}_m - \bar{\mathbf{x}}) \\ &= 1 + z\mathbf{E}\ \Psi' = \mathbf{E}\ (1 - z\Phi')^{-1}. \end{aligned}$$

Let Re $z \leq 0$ and $u > 0$. Then Re $zs(z) = r(z^{-1} - \Phi')^*$, where $r > 0$, and Re $zs(z) \leq 0$. Therefore, $|1 - zs(z)u| \geq 1$.

Let Im $z \neq 0$. The sign of z coincides with the sign of $h(z)$ and with the sign of $s(z)$. Therefore,

$$|1 - zs(z)u| = |z| \cdot |\text{Im }/|z|^2 + u\text{ Im } s(z)| \geq \alpha^{-1}.$$

This proves the lemma. □

THEOREM 2.2. *If $z \in \mathfrak{G}$ and $N > 1$, then*

$$\begin{aligned} &\text{Var }(n^{-1}\text{tr } H(z)) \leq a\tau^2\alpha^4/N, \\ &\mathbf{E}\ H(z) = (I - zs(z)\Sigma)^{-1} + \Omega, \\ &h(z) = n^{-1}\text{tr }(I - zs(z)\Sigma)^{-1} + \omega, \end{aligned} \tag{10}$$

where

$$\begin{aligned} &s(z) = 1 + y(h(z) - 1), \\ &\|\Omega\| \leq a\tau \max(1, \lambda)\ \alpha^3[\tau\alpha\sqrt{\delta} + (1 + \tau^2\alpha)/\sqrt{N}], \\ &|\omega| \leq a\tau\alpha^2 \max(1, \lambda)[\tau\alpha^2\sqrt{\delta} + (1 + \tau\alpha^3)/\sqrt{N}]. \end{aligned}$$

and a are numerical constants.

Proof. In the matrix C, let us single out the summand C^m independent of $\mathbf{x}_m$:

$$C = C^m + \Delta_m, \quad \Delta_m = (1 + N^{-1})\mathbf{x}_m\, \mathbf{x}_m^T - \mathbf{x}_m\, \bar{\mathbf{x}} - \bar{\mathbf{x}}\, \mathbf{x}_m^T].$$

Denote $H^m = (I - zC^m)^{-1}$. We have the identity $H^m = H + zH^m\Delta_m H$. Using Lemma 2.2, we obtain

$$\begin{aligned}\text{Var } (n^{-1}\text{tr } H) = \sum_{m=1}^{N} \mathbf{E}\ |zn^{-1}\text{tr } (H\Delta_m H^m)|^2 \\ \le 3|z|^2 n^{-2} N^{-1}\mathbf{E}\ ([(1 + N^{-1})|\mathbf{x}_m^T H^m H\mathbf{x}_m|^2 \\ +|\mathbf{x}_m^T H^m H\mathbf{x}_m|^2 + |\bar{\mathbf{x}}^T H^m H\bar{\mathbf{x}}_m|^2),\end{aligned}$$

where $\|H^m\| \le \alpha, \quad \|H\| \le \alpha, \quad \mathbf{E}\ (\mathbf{x}_m^2)^2 \le Mn^2, \quad$ and $\mathbf{E}\ |\bar{\mathbf{x}}|^4 \le My^2$. We find that Var $(n^{-1}\text{tr } H) \le 3\tau^2\alpha^4(1 + 3/N)/N$. The first statement of our theorem is proved.

Now, we start from Theorem 2.1. Obviously,

$$\begin{aligned}\mathbf{E}\ H =&(I - zs(z)\Sigma)^{-1} + \mathbf{E}\ H - \mathbf{E}\ H_0 + \\ &+ (I - zs(z)\Sigma)^{-1} z(s_0(z) - s(z))\Sigma(I - zs_0(z)\Sigma)^{-1} + \Omega_0.\end{aligned} \tag{11}$$

We have

$$|s(z) - s_0(z)| = y|h(t) - h_0(t)| \le \tau y\alpha^2/N.$$

Notice that the three last summands in the right hand side of (11) do not exceed $\tau^2\alpha^4 y/N + a\omega + \|\Omega_0\|$, where ω is from Lemma 2.5. Substituting these values, we obtain the required estimate of Ω. It follows

$$|\omega| \le \tau^2\alpha^4 y/N + \tau\alpha^2/N + a\tau^2\alpha^4(\sqrt{\delta} + \alpha/N).$$

This gives the required estimate of $|\omega|$. The proof is complete. □

Relations (10) were found in the form of limit formulas, first, in [41] for normal distributions. In [43] and [20] these limit expressions were derived for a wide class of populations. In [51] relations (10) were established for a wide class of distributions for fixed n and N.

Limit Spectral Functions of the Increasing Sample Covariance Matrices

We investigate here the limiting behaviour of spectral functions for the matrices S and C under the increasing dimension asymptotics. Consider a sequence $\mathfrak{P} = \{\mathfrak{P}_n\}$ of problems

$$\mathfrak{P}_n = (\mathfrak{S}, \Sigma, N, \mathfrak{X}, S, C)_n, \quad n = 1, 2, \ldots$$

in which spectral functions of matrices C and S are investigated over samples $\mathfrak{X}$ of size N from populations $\mathfrak{S}$ with $\operatorname{cov}(\mathbf{x}, \mathbf{x}) = \Sigma$ (we do not write out the subscripts for arguments of $\mathfrak{P}_n$). For each problem $\mathfrak{P}_n$, we consider functions

$$h_{0n}(t) = n^{-1}\mathrm{tr}\ (I - zS)^{-1}, \quad h_n(t) = n^{-1}\mathrm{tr}\ (I - zC)^{-1},$$

$$F_{0n}(u) = n^{-1}\sum_{i=1}^{n}\mathrm{ind}(\lambda_i^0 \le u), \quad F_n(u) = n^{-1}\sum_{i=1}^{n}\mathrm{ind}(\lambda_i \le u),$$

where λ_i^0 and λ_i are eigenvalues of S and C, respectively, $i = 1, \ldots, n$.

We restrict $\mathfrak{P}$ by the following conditions.

A. For each n the observation vectors in $\mathfrak{S}$ are such that $\mathbf{E}\ \mathbf{x} = 0$ and the four moments of all components of $\mathbf{x}$ exist.

B. The parameters (1) in $\mathfrak{P}$ do not exceed a constant c_0, where c_0 does not depend on n. The parameters (2) vanish as $n \to \infty$ in $\mathfrak{P}$.

C. In $\mathfrak{P}$, $n/N \to \lambda$.

D. In $\mathfrak{P}$ for each n, the eigenvalues of matrices Σ are located on a segment $[c_1, c_2]$, where $c_1 > 0$ and c_2 do not depend on n, and as $n \to \infty$, $F_{0n}(u) \to F_0(u)$ for each $u \ge 0$.

Corollary (of Theorem 2.2). Under assumptions A–D for each $z \in \mathfrak{G}$, the limit exists $\lim_{n\to\infty} h_n(z) = h(z)$ such that

$$h(z) = \int (1 - zs(z)u)^{-1} dF_0(u), \quad s(z) = 1 - \lambda + \lambda h(z), \qquad (12)$$

and for each z, we have

$$\lim_{n\to\infty} \|\mathbf{E}\ (I - zC)^{-1} - (I - zs(z)\Sigma)^{-1}\| \to 0.$$

Remark 7. Suppose that $n = 1, 2, \dots$ and $y = n/N \to \lambda > 0$ so that, for each n, there exists the moment $= \sup_{|\mathbf{e}|=1} \mathbf{E}\ (\mathbf{e}^T\mathbf{x})^6 < c$, where c does not depend on n. If (8) holds with $\omega \to 0$ as $n \to \infty$, then $\nu \to 0$.

Let us prove this assertion. Suppose that

$$\omega_n(z) \stackrel{\text{def}}{=} h(z) - n^{-1}\text{tr}\ (I - zs_0(z)\Sigma)^{-1} \to 0$$

uniformly in some neighborhood of the point $z = 0$. Near the point $z = 0$, the function $\omega_n(z)$ is thrice differentiable, and its third derivatives are uniformly bounded by a constant. At the point $z = 0$, we have $\omega_n(0) = 0$, $\omega_n'(0) = 0$, and, therefore, $\omega_n''(0) \to 0$. But

$$\omega_n''(0) = \mathbf{E}\ n^{-1}\text{tr}\ S^2 - \Lambda_2 - y\Lambda_1^2, \quad \text{where } \Lambda_i = n^{-1}\text{tr}\ \Sigma^i, \quad i = 1, 2.$$

On the other hand, by a straightforward calculation we obtain

$$\mathbf{E}\ n^{-1}\text{tr}\ S^2 = \Lambda_2(1 - N^{-1}) + y\Lambda_1^2 + y\ \text{var}(\mathbf{x}^2/n).$$

Consequently $\text{var}(\mathbf{x}^2/n) \to 0$. Now let Ω be any non-random symmetric positive semidefinite matrix with the spectral norm 1. We introduce new coordinates $\mathbf{x}' = \Omega^{1/2}\mathbf{x}$. In these coordinates, the new moment M_6' is such that $M_6' = \sup \mathbf{E}\ (\mathbf{e}^T\mathbf{x})^6 \le M_6$. By the arguments above we have

$$\text{var}(|\mathbf{x}'|^2/n) = \text{var}(\mathbf{x}^T\Omega\mathbf{x}/n) \to 0,$$

and it follows that $\nu \to 0$.

In the sense of Remark 7, the condition $\nu \to 0$ is not only sufficient for the limit formulas (12) to be true, but also necessary.

THEOREM 2.3. *If $h(z)$ satisfies (12), $c_1 > 0$, $\lambda > 0$, and $\lambda \ne 1$ for any $z \in \mathfrak{G}$, then*

1^o *$|h(z)| \le \alpha(z)$ and $h(z)$ is regular at any point $z \in \mathfrak{G}$;*

2^o *for any $v = \text{Re}\ z > 0$ such that $v < v_2 = c_1^{-1}(1 - \sqrt{\lambda})^{-2}$ or $v > v_1 = c_2^{-1}(1 + \sqrt{\lambda})^{-2}$, we have*

$$\lim_{\varepsilon \to +0} \text{Im}\ h(v + i\varepsilon) = 0;$$

3^o *if* $v_1 \le v \le v_2$, *then* $0 \le \text{Im } h(v+i\varepsilon) \le (c_1\lambda v)^{-1/2} + \omega$, *where* $\omega \to 0$ *as* $\varepsilon \to +0$.

4^o *if* $v = \text{Re } z < 0$ *then* $s(-v) \ge (1 + c_2\lambda|v|)^{-1}$;

5^o *if* $|z| \to \infty$ *on the main sheet of the analytical function* $h(z)$, *then we have*

$$\begin{aligned} &\text{if } 0 < \lambda < 1 \text{ then} \quad zh(z) = -(1-\lambda)^{-1}\Lambda_{-1} + O(|z|^{-1}), \\ &\text{if } \lambda = 1 \text{ then} \quad zh^2(z) = -\Lambda_{-1} + O(|z|^{-1/2}), \\ &\text{if } \lambda > 1 \text{ then} \quad zs(z) = -\beta_0 + O(|z|^{-1}). \end{aligned}$$

where β_0 *is a root of the equation*

$$\int (1+\beta_0 u)^{-1} dF_0(u) = 1 - \lambda^{-1}.$$

Proof. The existence of the solution to (12) follows from Theorem 2.2. Suppose Im $(z) > 0$. By Remark 5, $|h(z)| \le \alpha = \alpha(z)$. For brevity let $h = h(z),\ s = s(z)$. For all $u > 0$ and z outside the beam $z > 0$ we have $|1 - zsu|^{-1} \le \alpha$. Differentiating $h(z)$ in (12) we prove the regularity of $h(z)$.

Define

$$b_\nu = b_\nu(z) = \int |1 - zs(z)u|^{-2} u^\nu dF_0(u), \quad \nu = 1, 2.$$

Let us rewrite (12) in the form

$$(h-1)/s = z \int u(1 - zsu)^{-1} dF_0(u). \tag{13}$$

It follows that

$$\text{Im } [(h-1)/s] = |s|^{-2} \text{Im } h = b_1 \text{ Im } z + b_2\lambda|z|^2 \text{Im } h,$$

Dividing by b_2 we use the inequality $b_1/b_2 \le c_1^{-1}$. Let us fix some $v = \text{Re } z > 0$ and tend Im $z = \varepsilon \to +0$. It follows that $(|s|^{-2}b_2^{-1} - \lambda v^2)$ Im $h \to 0$. Suppose that Im h does not tend to 0 (v is fixed). Then there exits a sequence $\{z_k\}$ such that, for $z_k = v + i\varepsilon_k$, $h =$

$h(z_k)$, $s = s(z_k)$, we have Im $h \to a$, where $a \neq 0$. For these z_k, we obtain $|s|^{-2}b_2^{-1} \to \lambda v^2$ as $\varepsilon_k \to +0$. We apply the Cauchy–Bunyakovskii inequality to (13). It follows that $|h-1|^2/|z_k s|^2 \leq b_2$. We obtain that $|h-1|^2 \leq \lambda^{-1} + o(1)$ as $\varepsilon_k \to +0$. It follows that $|s-1|^2 \leq \lambda + o(1)$. So the values s are bounded for $\{z_k\}$. On the other hand, it follows from (12) that Im $h = b_1$ Im $(zs) = b_1$(Re $s \cdot$ Im $z + \lambda v$ Im h). We find that $(b_1^{-1} - \lambda v)$ Im $h \to 0$ as Im $z \to 0$. But Im $h \to a \neq 0$ for $\{z_k\}$. It follows that $b_1^{-1} \to \lambda v$. Combining this with the inequality $|s|^{-2}b_2^{-1} \to \lambda v^2$ we find that $|s|^{-2}b_2^{-1} - b_1^{-1}v \to 0$. Note that b_1 is finite for $\{z_k\}$ and $c_2^{-2} \leq b_1 b_2^{-1} \leq c_1^{-1}$. Substitute the boundaries $(1 \pm \sqrt{\lambda}) + o(1)$ for $|s|$. We obtain that $v_1 + o(1) \leq v \leq v_2 + o(1)$ as $\varepsilon_k \to +0$. We conclude that Im $h \to 0$ for any positive v outside the interval $[v_1, v_2]$. This proves the second statement of our theorem.

Now suppose $v_1 \leq v \leq v_2$. From (12) we obtain the equality $\lambda(\text{Im } h)\text{Im } (zh) \leq c_1^{-1}$. But h is bounded. The inequality $(\text{Im } h)^2 \leq (c_1 v\lambda)^{-1}$ follows. The third statement of our theorem is proved.

Further, let $v = \text{Re } z < 0$. Then the functions h and s are real and non-negative. We multiply both parts of (12) by λ. It follows that $(h-1)/zs \leq b_1 \leq c_2$. We obtain $s \geq (1 + c_2\lambda|z|)^{-1}$.

Let us prove the fifth theorem statement. Let $\lambda < 1$. For real $z \to -\infty$, the value Re $(1 - zsu)$ in the integrand of (12) tends to infinity. Consequently $h \to 0$ and $s \to 1 - \lambda$. For sufficiently large $|\text{Re } z|$, we have

$$h(z) = \sum_{k=1}^{\infty} \Lambda_{-k}(zs)^{-k},$$

where $\Lambda_k = \int u^k dF_0(u)$. We conclude that

$$h(z) = -(1-\lambda)^{-1}\Lambda_{-1}z^{-1} + O(|z|^{-2})$$

for real $z < 0$ and for any $z \in \mathfrak{G}$ as $|z| \to \infty$ in view of the properties of the Laurent series. Now let $\lambda = 1$. Then $h = s$. From (12) we obtain that $h \to 0$ as $z \to -\infty$ and $h^2 = \Lambda_{-1}|z|^{-1} + O(|z|^{-2})$. Suppose $\lambda > 1$, $z = -t < 0$, and $t \to \infty$. It follows that $|h(1 + tc_1 s)| \leq 1$. One can see that h cannot be arbitrary small. Then the product ts is finite, $s \to 0$, and $h \to 1 - \lambda^{-1}$. From (12) we obtain $ts \to \beta_0$ as is stated in the theorem formulation. This completes the proof of Theorem 2.3. □

Remark 8. Under assumptions A–D for each $u \geq 0$, the limit exists

$$F(u) = \operatorname*{plim}_{n\to\infty} F_n(u)$$

$$\text{such that} \quad \int (1 - zu)^{-1} dF(u) = h(z). \tag{14}$$

To prove the convergence we can cite Lemma 1 of the Introduction which assumes the convergence of $\{h_{0n}(z)\}$ and $\{h_n(z)\}$ in probability. By Lemma 2.5 both the sequences converge to the same limit $h(z)$. The both relations (14) follow. To prove that the limits of $F_{0n}(u)$ and $F_n(u)$ coincide it suffices to prove the uniqueness of the solution to (12). It can be readily proved if we perform the inverse Stieltjes transformation.

THEOREM 2.4. *Under assumptions A–D,*

1^0 *if* $\lambda = 0$, *then* $F(u) = F_0(u)$ *almost everywhere for* $u \geq 0$*;*

2^0 *if* $\lambda > 0$ *and* $\lambda \neq 1$ *then* $F(0) = F(u_1 - 0) = \max(0, 1 - \lambda^{-1})$, $F(u_2) = 1$, *where* $u_1 = c_1(1 - \sqrt{\lambda})^2$, $u_2 = c_2(1 + \sqrt{\lambda})^2$, *and* c_1 *and* c_2 *are bounds of the limit spectra of* Σ.

3^0 *if* $y > 0$, $\lambda \neq 1$, *and* $u > 0$, *then the derivative* $F'(u)$ *of the function* $F(u)$ *exists and* $F'(u) \leq \pi^{-1}(c_1\lambda u)^{-1/2}$*;*

Proof. Let $\lambda = 0$. Then $s(z) = 1$. By (12) and (13) we have

$$h(z) = \int (1 - zu)^{-1} dF(u) = \int (1 - zu)^{-1} dF_0(u).$$

At the continuity points of $F_0(u)$ the derivative

$$F_0'(u) = \frac{1}{\pi} \lim_{\varepsilon \to +0} \operatorname{Im} \frac{1}{z} h\left(\frac{1}{z}\right) = F'(u),$$

where $z = u - i\varepsilon$, $u > 0$.

Let $\lambda > 0$. By Theorem 2.2 for $u < u_1$ and for $u > u_2$ (note that $u_1 > 0$ if $\lambda > 0$) the values Im $[(u - i\varepsilon)^{-1} h((u - i\varepsilon)^{-1})] \to 0$ as $\varepsilon \to +0$. But we have

$$\text{Im } \frac{h((u - i\varepsilon)^{-1})}{u - i\varepsilon} > (2\varepsilon)^{-1}[F(u + \varepsilon) - F(u - \varepsilon)]. \qquad (15)$$

It follows that $F'(u)$ exists and $F'(u) = 0$ for $0 < u < u_1$ and for $u > u_2$. The points of increase of $F(u)$ can be located only at the point $u = 0$ or on the segment $[u_1, u_2]$. If $\lambda < 1$ and $|z| \to \infty$, we have $\int (1 - zu)^{-1} dF(u) \to 0$ and, consequently, $F(0) = 0$. If $y > 1$ and $|z| \to \infty$ then $h(z^{-1})z^{-1} \approx (1 - \lambda^{-1})/z$ and $F(0) = 1 - \lambda^{-1}$. The second statement of our theorem is proved.

Now, let $z = v + i\varepsilon$, where $v > 0$ is fixed and $\varepsilon \to +0$. Then from (12), we obtain that Im $h = b_1$Im (zs). Obviously,

$$|\text{Im } h| \le \int |1 - zsu|^{-1} dF_0(u) \le \frac{1}{c_1 \text{Im } (zs)} = \frac{b_1}{c_1 \text{Im } h}.$$

If Im h remains finite, then $b_1 \to (\lambda v)^{-1}$. Using (15), we obtain the last statement of the theorem. Theorem 2.3 is proved. □

THEOREM 2.5. *Under conditions A–D if* $0 < \lambda < 1$ *then for complex* z, z' *outside of the half-axis* $z > 0$

$$|h(z) - h(z')| < c_3 |z - z'|^{\zeta},$$

where c_3 *and* $\zeta > 0$ *do not depend on* z *and* z'.

Proof. From (12) it follows

$$|h(z)| \le \lambda^{-1} \max\ (\lambda, |1 - \lambda| + 2c_1^{-1}|z|^{-1}).$$

By Theorem 2.3 the function $h(z)$ is differentiable for each z outside the segment $\mathfrak{V} = [v_1, v_2]$, $v_1 > 0$.

Denote a δ-neighbourhood of the segment $\mathfrak{V}$ by $\mathfrak{V}_\delta$. If z is outside of $\mathfrak{V}_\delta$, then the derivative $h'(z)$ exists and is uniformly bounded. It suffices to prove our theorem for $v \in \mathfrak{V}_1$, where $\mathfrak{V}_1 = \mathfrak{V}_\delta - \{z : \text{Im } z = 0\}$. Choose $\delta = \delta_1 = v_1/2$. Then $\delta_1 < |z| < \delta_2$ for $z \in \mathfrak{V}_1$, where δ_2 does not depend on z. Let us estimate the absolute value of the derivative $h'(z)$. For Im $z \neq 0$ from (12), we obtain by the differentiation that

$$\left(z^{-1}y^{-1} - \int X^{-2}udF_0(u)\right)h'(z) = \frac{s(z)}{z\lambda}\int X^{-2}udF_0(u), \quad (16)$$

where $X = (1 - zs(z)u) \neq 0$ by Lemma 2.6.

Denote

$$\varphi(z) = \frac{1}{zy} - \int X^{-2}udF_0(u), \quad b_1 = \int |X|^{-2}udF_0(u),$$
$$h_1 = \text{Im } h(z), \quad z_0 = \text{Re } z, \quad z_1 = \text{Im } z, \quad s_0 = \text{Re } s(z),$$

and let α with subscripts denote constants not depending on z. The right hand side of (16) is not greater $\alpha_1 b_1$ for $z \in \mathfrak{V}_1$, and therefore, $|h'(z)| < \alpha_2 b_1 \, |\varphi(z)|^{-1}$. We consider two cases. Denote $\alpha_3 = (2\delta_2 c_2)^{-1}$.

At first, let $s_0 \leq \alpha_3$. Using the relation $h_1 = b_1$ Im $(zs(z))$, we obtain that the value $-$Im $\varphi(z)$ equals

$$z_1|z|^{-2}\lambda^{-1} + 2b_1{}^{-1}\int |X|^{-4}u^2(1 - z_0 s_0 u + z_1\lambda h_1 u)h_1 dF_0(u).$$

In the integrand here, we have $z_0 > 0$, $1 - z_0 s_0 u \geq 1/2$, $z_1 h_1 > 0$. From the Cauchy–Bunyakovskii inequality it follows that

$$\int |X|^{-4}u^2 dF_0(u) \geq b_1^2.$$

Hence $|\text{Im } \varphi(z)| \geq b_1 h_1$ and $|h'(z)| \leq \alpha_2 h_1^{-1}$. Let

$$\text{Re } \varphi(z) = \lambda^{-1}z_0|z|^{-2} - b_1 + 2\,[\text{Im } zs(z)]^2 \int |X|^{-4}u^3 dF_0(u).$$

Define $p = \lambda^{-1}z_0|z|^{-2} - b_1$. We have

$$p = \lambda^{-1}|z|^{-2}z_0 z_1 \; |\text{Im } zs(z)|^{-1}(s_0 - \lambda h_1 z_1/z_0).$$

Here $|h_1| < \alpha_4$, $z_0 \geq \delta_1 > 0$, $s_0 > \alpha_3 > 0$, and we obtain that $p > 0$ if $z_1 < \alpha_6$, where $\alpha_6 = \alpha_3\alpha_5/\lambda\alpha_4$. If $z \in \mathfrak{V}_1$ and $z_1 > \alpha_6$, then the

Hölder inequality follows from the existence of a uniformly bounded derivative of the analytic function $h(z)$ in a closed domain.

Now let $z \in \mathfrak{V}_1$, $z_1 < \alpha_6$, $p > 0$, and $s_0 > \alpha_3 > 0$. Then $|h'(z)| \le \alpha_7 b_1 |\text{Re } \varphi(z)|^{-1}$, where

$$\begin{aligned}\text{Re } \varphi(z) &\ge 2(\text{Im } zs(z))^2 c_1 \int |X|^{-4} u^2 dF_0(u) \\ &\ge 2(\text{Im } zs(z))^2 c_1 b_1 = 2c_1 h_1^2.\end{aligned}$$

Substituting $b_1 = h_1/\text{Im } (zs(z))$ and taking into account that $s_0 > 0$, we obtain that $|h'(z)| \le \alpha_7 h_1^{-2}$. Thus for $v \in \mathfrak{V}_\delta$ and $0 < z_1 < \alpha_6$ for any s_0, it follows that $|h'(z)| \le \alpha_8 \max\,(h_1^{-1}, h_1^{-2}) \le \alpha_9 h_1^{-2}$. Taking the derivative along the vertical line we obtain the inequality $h_1^2 |dh_1/dz| \le \alpha_9$, whence

$$h_1^3(z) \le h_1^3(z') + 3\alpha_9 |z - z'| \le \Big(h_1(z') + \alpha_{10} |z - z'|^{1/3}\Big)^3$$

if $\text{Im } z \cdot \text{Im } z' > 0$. The Hölder inequality follows for $h_1 = \text{Im } h(z)$ with $\zeta = 1/3$. This completes the proof of Theorem 2.5. □

Example. Let us consider limit spectra of matrices Σ of a special form of the 'ρ-model' considered first in [41]. It is of a special interest since it admits an analytical solution to the equation (12). For this model, the limit spectrum of Σ is located on a segment $[c_1, c_2]$, where $c_1 = \sigma^2(1 - \sqrt{\rho})^2$ and $c_2 = \sigma^2(1 + \sqrt{\rho})^2$, $\sigma > 0$,
$0 \le \rho < 1$. Its limit spectrum density is

$$\frac{dF_0(u)}{du} = \begin{cases} (2\pi\rho)^{-1}(1-\rho)u^{-2}\sqrt{(c_2 - u)(u - c_1)}, & c_1 \le u \le c_2, \\ 0 \quad \text{for} \quad u < c_1 \quad \text{and for} \quad u > c_2. \end{cases}$$

The moments $\Lambda_k = \int u^k dF_0(u)$ for $k = 0, 1, 2, 3, 4$ are

$$\begin{gathered}\Lambda_0 = 1, \quad \Lambda_1 = \sigma^2(1-\rho), \quad \Lambda_2 = \sigma^4(1-\rho), \\ \Lambda_3 = \sigma^6(1-\rho^2), \quad \Lambda_4 = \sigma^8(1-\rho)(1+3\rho+\rho^2).\end{gathered}$$

If $\rho > 0$, the integral

$$\eta(z) = \int (1 - zu)^{-1} dF_0(u) = \frac{1 + \rho - \kappa z - \sqrt{(1 + \rho - \kappa z)^2 - 4\rho}}{2\rho},$$

where $\kappa = \sigma^2(1-\rho)^2$. The function $\eta = \eta(z)$ satisfies the equation $\rho\eta^2 + (\kappa z - \rho - 1)\eta + 1 = 0$. The equation $h(z) = \eta(zs(z))$ can be transformed to the equation $(h-1)(1-\rho h) = \kappa z h s$ which is quadratic with respect to $h = h(z)$, $s = 1 - \lambda + \lambda h$. If $\lambda > 0$ its solution is

$$h = \frac{1+\rho-\kappa(1-\lambda)z - \sqrt{(1+\rho-\kappa(1-\lambda)z)^2 - 4(\rho+\kappa z\lambda)}}{2(\rho+\kappa\lambda z)}.$$

The moments $M_k = (k!)^{-1}h^{(k)}(0)$ for $k = 0, 1, 2, 3$ are

$$M_0 = 1,\quad M_1 = \sigma^2(1-\rho),\quad M_2 = \sigma^4(1-\rho)\,(1+\lambda(1-\rho))\,,$$
$$M_3 = \sigma^6(1-\rho)\left(1+\rho+3\lambda(1-\rho)+\lambda^2(1-\rho)^2\right).$$

Differentiating the functions of the inverse argument, we find that, in particular, $\Lambda_{-1} = \kappa^{-1}$, $\Lambda_{-2} = \kappa^{-2}(1+\rho)$, $M_{-1} = \kappa^{-1}(1-\lambda)^{-1}$, $M_{-2} = \kappa^{-2}\,(\rho+\lambda(1-\rho))\,(1-\lambda)^{-3}$. The continuous limit spectrum of the matrices C is located on the segment $[u_1, u_2]$, where

$$u_1 = \sigma^2(1-\sqrt{\lambda+\rho(1-\lambda)})^2,\quad u_2 = \sigma^2(1+\sqrt{\lambda+\rho(1-\lambda)})^2$$

and has the density

$$f(u) = \begin{cases} \dfrac{(1-\rho)\sqrt{(u_2-u)(u-u_1)}}{2\pi u(\rho u + \sigma^2(1-\rho)^2 y)} & \text{if } u \in [u_1, u_2], \\ 0 \quad \text{otherwise.} & \end{cases}$$

If $\lambda > 1$ then the function $F(u)$ has a jump $1-\lambda^{-1}$ at the point $u = 0$. If $\lambda = 0$ then $F(u) = F_0(u)$ has a form of a unit step at the point $u = \sigma^2$. In the special case when $\rho = 0$ and $\lambda \neq 0$ we obtain the functions $h(z)$ and $f(u)$ which were found in Chapter 1. The density $f(u)$ satisfies the Hölder condition with $\zeta = 1/2$.