CHAPTER VI

EPSILON-DOMINATING COMPONENT-WISE SHRINKAGE ESTIMATORS OF NORMAL MEAN

In this chapter we investigate the effect of an improved component-wise estimation of the expectation vectors for normal vectors with independent components. The dimension of variables and the sample size are supposed to be sufficiently high to apply the technique of singling out the leading terms in the asymptotics of the increasing dimension. But we will not pass to the limit and will obtain relations valid for any chosen dimension and any chosen sample size along with upper estimates of the remainder terms accurate up to absolute constants.

Denote the observation vector by $\mathbf{x} = (x_1, \ldots, x_n)$, and let $\mu = (\mu_1, \ldots, \mu_n) = \mathbf{E}\ \mathbf{x}$ be the vector of parameters. Suppose that $\mathbf{x} \sim \mathbf{N}(\mu, I)$, where I is the identity matrix. Let $\bar{\mathrm{x}} = (\bar{x}_1, \ldots, \bar{x}_n)$ be a sample mean vector calculated over a sample of size N. The quadratic risk of the estimator $\bar{\mathbf{x}}$ is, obviously, $\mathbf{E}\ (\mu - \bar{\mathbf{x}})^2 = n/N$ (here and in the following, squares of vectors denote squares of lengths).

Remark 1. The Stein estimator $\mu^S = (1 - (n-2)/N\bar{\mathbf{x}}^2)\bar{\mathbf{x}}$ has the following extremum property for $n > 2$:

$$\mathbf{E}\ (\mu - \mu^S)^2 \leq \min_{\eta} \mathbf{E}\ (\mu - \eta\bar{\mathbf{x}})^2 + \frac{4(n-1)}{nN}\ ,$$

where η is a non-random scalar. The extremum value of η is equal to $\eta^{\mathrm{opt}} = \mu^2(\mu^2 + n/N)^{-1}$ and for this η, the first summand in the right hand side of (1) equals $\eta^{\mathrm{opt}} n/N$.

Estimation Function for the Component-Wise Estimators

To investigate the effect of component-wise estimation of the expectation value vectors, we consider a class $\mathfrak{K}_0$ of estimators μ of

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX the form $\widehat{\mu} = (\varphi(\bar{x}_i),\ i = 1, \dots, n)$, where the 'estimation function' $\varphi(t)$ is a non-random measurable function of bounded increase, $|\varphi(t)| \le c|t|$, where c does not depend on t. Let us minimize $R = R(\varphi) = \mathbf{E}\ (\mu - \widehat{\mu})^2$.

We use the basic approach of the increasing dimension asymptotics: to consider sets of variables in the form of functions of their empiric distribution. Denote

$$\widehat{F}_0(t) = n^{-1} \sum_i \operatorname{ind}(\bar{x}_i \le t),$$
$$F_0(t) = \mathbf{E}\ \widehat{F}_0(t) = n^{-1} \sum_i \Phi\left(\sqrt{N}(t - \mu_i)\right)$$

(here and in the following, $i = 1, \dots, n$ in the sums). Denote

$$f_0(t) = \frac{F_0(t)}{dt} = n^{-1} \sqrt{\frac{N}{2\pi}} \sum_i \exp\left(-\frac{N(t - \mu_i)^2}{2}\right).$$

Remark 2.

The function $\varphi(t) = \varphi^{\text{opt}}(t) = t + N^{-1}\partial/\partial t\ \ln f_0(t)$ minimizes the quadratic risk of estimators $\widehat{\mu}$ from $\mathfrak{K}_0$ so that for any $\varphi(\cdot)$, $R = R_{\text{opt}} = R(\varphi^{\text{opt}}) \le R(\varphi)$, where

$$R(\varphi^{\text{opt}}) = \frac{n}{N} - \frac{n}{N^2} \int \left(\frac{\partial}{\partial t} \ln f_0(t)\right)^2 f_0(t)\, dt.$$

Note that in a special case when $\mu_1 = \mu_2 = \dots = \mu_n$, the quadratic risk $R_{\text{opt}} = 0$.

To estimate $f_0(t)$ using $\widehat{F}_0(t)$, a smoothing is necessary. For a more convenient isolation of leading terms, we change the scale and pass to an equivalent problem of estimating vectors $\mathbf{v} = (v_1, \dots, v_n)$ of the parameters $v_i = \sqrt{n}\mu_i$ using the statistics $\mathbf{u} = (u_1, \dots, u_n)$, where $u_i = \sqrt{n}\bar{x}_i$, $i = 1, \dots, n$. We consider the class $\mathfrak{K}_1$ of estimators of $\mathbf{v}$

$$\widehat{\text{v}} = (\varphi(u_i),\ i = 1, \dots, n), \tag{1}$$

where $|\varphi(t)| \le c|t|$ for all t, and c does not depend on t.

Denote

$$y = n/N, \quad R(\varphi) = n^{-1}\mathbf{E}\ (\widehat{\mathbf{v}} - \mathbf{v})^2,$$

$$f(t,d) = n^{-1}\frac{1}{\sqrt{2\pi d}}\sum_i \exp\,(-(t-v_i)/2d), \quad d>0.$$

Remark 3. The following identities hold:

$$R(\varphi) = R_{\rm opt} + \int [\varphi(t) - \varphi^{\rm opt}(t,y)]^2 f(t,y)\,dt,$$

$$\text{where} \quad R_{\rm opt} = R(\varphi^{\rm opt}) = y - y^2 \int \Big[\frac{\partial}{\partial t}\ln\, f(t,y)\Big]^2 f(t,y)\,dt,$$

$$\text{and} \quad \varphi^{\rm opt}(t,y) = t + y\frac{\partial}{\partial t}\ln\, f(t,y).$$

Thus, the unknown estimator

$$\mathbf{v}^{\rm opt} = (\varphi^{\rm opt}(u_1,y),\dots,\varphi^{\rm opt}(u_n,y)) \tag{2}$$

is dominating over the class $\mathfrak{K}_1$ with respect to the quadratic risk function.

Estimators of the Unimprovable Estimation Function

To obtain an estimator for $\mathrm{v}^{\rm opt}$, we consider the statistic

$$\widehat{f}(t) = \widehat{f}(t,\varepsilon) = n^{-1}\sum_i\,(2\pi\varepsilon)^{-1/2}\exp\,\Big(-\frac{(t-u_i)^2}{2\varepsilon}\Big), \quad \varepsilon>0.$$

LEMMA 6.1. *For any $\varepsilon > 0$,*

$$\mathbf{E}\;\widehat{f}(t) = f(t,y+\varepsilon), \quad \mathrm{var}\;\widehat{f}(t) \le n^{-1}(2\pi\varepsilon)^{-1}f(t,d),$$

where $d = y + \varepsilon/2$.

Proof. We note that for an arbitrary bounded function $\psi(t)$, we have

$$\sum_i \mathbf{E}\;\psi(u_i) = \int \psi(t)\,f(t,y)\,dt.$$

To prove this lemma it suffices to examine the unbiasedness of the estimator $\widehat{f}(t,\varepsilon)$ of $f(t,y+\varepsilon)$ by the integration and examine that the variance of $\widehat{f}(t,\varepsilon)$ equals

$$n^{-2}(2\pi\varepsilon)^{-1}\sum_i \text{var}\ \big(\exp\,(-(t-u_i)^2/2\varepsilon)\big)$$
$$\le n^{-2}(2\pi\varepsilon)^{-1}\sum_i \mathbf{E}\exp\,(-(t-u_i)^2/\varepsilon)$$
$$= n^{-2}(4\pi\varepsilon)^{-1/2}(2\pi d)^{-1/2}\sum_i \exp\,(-(t-v_i)^2/2d),$$

where $d = y+\varepsilon/2,\ \ \varepsilon>0$. This proves Lemma 6.1. □

To construct an estimator of $\varphi^{\text{opt}}(t,y)$ with the bounded increase, it is necessary to restrict the decrease of the estimators of $f(t,\varepsilon)$ for large t. We consider the statistics

$$\widehat{\varphi}^{\text{opt}}(t,\varepsilon,\delta) = t + \text{ind}\,(\widehat{f}(t,\varepsilon)>\delta)\ y\ \frac{\partial}{\partial t}\ln\,\widehat{f}(t,\varepsilon),\ \varepsilon,\delta>0,$$
$$\widehat{\mathbf{v}}^{\text{opt}} = \big(\widehat{\varphi}(u_1,\varepsilon,\delta),\ldots,\widehat{\varphi}(u_n,\varepsilon,\delta)\big). \qquad (3)$$

We prove that the square risk of the estimator (3) of v approaches to the quadratic risk R_{opt} of the estimator (2) for large n and some ε and δ. For this purpose we estimate the increase of the quadratic risk when we pass from $\varphi^{\text{opt}}(t,y)$ to $\varphi^{\text{opt}}(t,y+\varepsilon)$, and then from $\varphi^{\text{opt}}(t,y+\varepsilon)$ to $\widehat{\varphi}^{\text{opt}}(t,\varepsilon,\delta)$. Denote

$$\rho_1 = \mathbf{E}\ n^{-1}\sum_i \big[v_i-\varphi^{\text{opt}}(u_i,y)\big]^2,$$
$$\rho_2 = \mathbf{E}\ n^{-1}\sum_i \big[\varphi^{\text{opt}}(u_i,y)-\varphi^{\text{opt}}(u_i,y+\varepsilon)\big]^2,$$
$$\rho_3 = \mathbf{E}\ n^{-1}\sum_i \big[\varphi^{\text{opt}}(u_i,y+\varepsilon)-\widehat{\varphi}^{\text{opt}}(u_i,\varepsilon,\delta)\big]^2.$$

We need the following two lemmas.

LEMMA 6.2. *Let $\varepsilon>0$. The quantity $\rho_2\le a\varepsilon^2/y$, where a is a numerical constant.*

To be concise, denote $x_i = |v_i-t|/d,\ \ d>0$,

$$f_i = \frac{1}{\sqrt{2\pi d}}\ \exp\Big(-\frac{(t-v_i)^2}{2d}\Big),$$
$$f_{it} = \frac{\partial}{\partial t}f_i,\quad f_{id} = \frac{\partial}{\partial d}f_i,\quad f_{itd} = \frac{\partial^2}{\partial d\,\partial t}f_i.$$

Let us express ρ_2 in terms of derivatives of function $\varphi^{\text{opt}}(\cdot)$ at some intermediate value $d_i\colon y \le d_i \le y + \varepsilon$. We find that

$$\rho_2 = n^{-1}\varepsilon^2 \mathbf{E} \sum_i \Big[\frac{\partial \varphi^{\text{opt}}(u_i, d)}{\partial d}\Big]^2 =$$
$$= n^{-1}\varepsilon^2 \mathbf{E} \sum_i \Big(\frac{|f_{it}|}{f_i} + \frac{|f_{itd}|}{f_i}\, d_i + \frac{|f_{it} f_{id}|}{f_i^2}\, d_i\Big)^2,$$

where the arguments of f_i and its derivatives are $d = d_i = d_i(u_i)$ for each i under the sign of summation. The calculation of expectation values for each i is reduced to the integration with respect to u_i with the weight

$$(2\pi y)^{-1/2} \exp\,(-(u_i - v_i)^2/2y) \le \sqrt{d_i/y}\, f_i.$$

Keeping only the squares of summands, we obtain

$$\rho_2 \le 3n^{-1}\varepsilon^2 \mathbf{E} \sum_i \sqrt{\frac{d_i}{y}} \,\left(\frac{f_{it}^2}{f_i} + \frac{f_{itd}^2}{f_i}\, d_i^2 + \frac{f_{it}^2 f_{id}^2}{f_i^3}\, d_i^2\right),$$

where the arguments are $d = d_i$ for each i.

Denote $x_i = (v_i - u_i)/\sqrt{d_i}, \quad i = 1, \ldots, n$. We find that

$$f_{it}^2 \le x_i^2 f_i^2/d_i \le (x_i^2 + 1)^2 f_i^2/4d_i, \quad f_{itd}^2 \le (x_i^2 + 2|x_i| + 1)^2 f_i^2/4d_i^3.$$

Therefore,

$$\rho_2 \le a\varepsilon^2 \mathbf{E}\, n^{-1} \sum_i \frac{1}{\sqrt{d_i y}}\,(x_i^2 + x_i^6)\, f_i$$
$$\le b\varepsilon^2 y^{-3/2} n^{-1} \sum_i \mathbf{E}\,\, (x_i^2 + x_i^6) \exp\,(-x_i^2/2).$$

Here a and b are numerical constants. Each term under the sign of expectation in the last inequality does not exceed a constant. We conclude that $\rho_2 \le a\varepsilon^2 y^{-1}$. This proves Lemma 6.2. □

Let us split ρ_3 into two summands

$$\rho_{31} = \mathbf{E}\; n^{-1} \sum_i [\varphi^{\text{opt}}(u_i, y + \varepsilon) - \widehat{\varphi}^{\text{opt}}(u_i, \varepsilon, \delta)]^2 \operatorname{ind}(\widehat{f}(u_i) > \delta),$$
$$\text{and} \quad \rho_{32} = d^2 n^{-1} \mathbf{E} \sum_i r^2(u_i) \operatorname{ind}(\widehat{f}(u_i) \le \delta),$$

where $r(t) = r(t, d) = \partial \ln\, f(t, d)/\partial t, \quad d = y + \varepsilon.$

LEMMA 6.3. *If* $0 < \varepsilon \le y$ *and* $n > 1$, *then we have* $\rho_{31} \le ayn^{-1}\lambda^{-2}(y/\varepsilon)^{3/2}$, *where* $\lambda = \delta\sqrt{2\pi y}$ *and the coefficient* a *is a numerical constant.*

Proof. Denote

$$\widehat{f}_i(t) = (2\pi\varepsilon)^{-1/2}\exp\,(-(t-u_i)^2/2\varepsilon),$$
$$\xi_i(t) = r(t,d)\,\widehat{f}_i(t) - \frac{d}{dt}\,\widehat{f}_i(t) = \widehat{f}_i(t)\left[r(t,d) + \frac{t-u_i}{\varepsilon}\right],$$
$$i = 1\ldots,n, \quad q(t) = n^{-1}\sum_i \xi_i(t).$$

We find that

$$\rho_{31} \le y^2\delta^{-2}\mathbf{E}\;n^{-1}\sum_i\left[r(u_i,d)\,\widehat{f}(u_i) - \widehat{f}'(u_i)\right]^2\mathrm{ind}\,(\widehat{f}^i(u_i) > \delta)$$
$$\le y^2\delta^{-2}n^{-1}\sum_i \mathbf{E}\;q^2(u_i)$$

(here and in the following, primes denote derivatives with respect to an explicit argument). For any non-random t, we have

$$\mathbf{E}\;q(t) = r(t,d)\,f(t,d) - \frac{\partial\,f(t,d)}{\partial t} = 0.$$

Consequently,

$$\mathbf{E}\;q^2(t) = \mathrm{var}\;q(t) = n^{-2}\sum_i \mathrm{var}\;\xi_i(t).$$

Further, $\mathbf{E}\;\widehat{f}_i^2(t)$ equals

$$(2\pi\varepsilon)^{-1}(2\pi y)^{-1/2}\int \exp\,[-(t-z)^2/\varepsilon - (z-v_i)^2/2y]\,dz =$$
$$= (2\pi)^{-1}\varepsilon^{-1/2}\tilde{d}^{-1/2}\,\exp\,(-(t-v_i)^2/2\tilde{d})\,, \quad i = 1,\ldots,n,$$

where $\tilde{d} = y + \varepsilon/2$. It follows that, for any t and i, this quantity is not greater $(2\pi)^{-1/2}(y\varepsilon)^{-1/2}$. We obtain that

$$\mathbf{E}\;[f_i'(t)]^2 = \frac{1}{2\pi\varepsilon}\int \frac{1}{\sqrt{2\pi y}}\exp\,[-\frac{(t-z)^2}{\varepsilon} - \frac{(z-v_i)^2}{2y}]\;\frac{(t-z)^2}{\varepsilon^2}\;dz$$
$$\le (4\pi\sqrt{2})^{-1}y^{-1/2}\varepsilon^{-3/2}, \quad i = 1,\ldots,n.$$

For any $i = 1, \dots, n$, we have

$$\mathbf{E}\, \xi_i^2(t) < \pi^{-1}\varepsilon^{-1/2}y^{-1/2}[r^2(t,d) + \varepsilon^{-1}], \quad d = y + \varepsilon, \quad \varepsilon > 0.$$

We single out the dependence on u_i: let $\mathbf{E} = \mathbf{E}_i\mathbf{E}^i$, where the expectation $\mathbf{E}_i$ is calculated by the integration only with respect to the distribution of u_i, whereas $\mathbf{E}^i$ is calculated over the distribution of the remaining u_j, $j \neq i$. Obviously,

$$\mathbf{E}^i q(t) = n^{-1}(\xi_i(t) - \mathbf{E}_i\xi_i(t)),$$
$$\mathbf{E}^i q^2(t) = n^{-2}\Big[\sum_{j\neq i} \operatorname{var}\, \xi_j(t) + \big(\xi_j(t) - \mathbf{E}_i\xi_j(t)\big)\Big], \quad i = 1, \dots, n.$$

Let us prove that the second summand in the square brackets is only a correction to the first one. Note that the quantity $\xi_i(u_i) = (2\pi\varepsilon)^{-1/2}r(u_i, d)$, and, consequently, $|\mathbf{E}_i\xi_i(t)|$ equals

$$\frac{1}{(2\pi d)^{-1/2}} \exp\Big(-\frac{(t-v_i)^2}{2d}\Big)\Big|r(t,d) + \frac{t-v_i}{d}\Big| \le \frac{r(t,d) + d^{-1/2}}{\sqrt{2\pi d}}$$

for any non-random argument $t > 0$, $i = 1, \dots, n$. We obtain the inequality

$$\mathbf{E}^i q^2(u_i) \le n^{-2}s(u_i), \quad i = 1, \dots, n,$$
$$\text{where } s(t) = \sum_j \operatorname{var}\, \xi_j(t) + r^2(t,d)/\varepsilon + d^{-2}.$$

It follows that

$$\frac{1}{n}\sum_i \mathbf{E}q^2(u_i) = \frac{1}{n}\sum_i \mathbf{E}_i[\mathbf{E}^{\,i}q(t)]_{t=u_i} \le \frac{1}{n}\int s(t)\, f(t,y)\, dt.$$

Now we majorize the upper estimate of $\operatorname{var}\, \xi_j(t)$ with the second moment and use Remark 3 (for $y = d$). It follows that (for $\varepsilon < y$), the right hand side is not larger than

$$\pi^{-1}n^{-1}\varepsilon^{-3/2}y^{-1/2}(1 + \theta + 2\pi\theta^{1/2}n^{-1}), \quad \text{where } \theta = \varepsilon/y.$$

The proof of Lemma 6.3 is complete. □

Lemma 6.4. *If* $0 < \varepsilon \le y$, *then for any positive* $\alpha < 1$,

$$\frac{\rho_{32}}{2} \le yn^{-1}\lambda^{-2}\Big(\frac{y}{\varepsilon}\Big)^{1/2} + a\ \frac{y\lambda^{\alpha}}{(1-\alpha)^{3/2}},$$

where $\lambda = \delta\sqrt{2\pi y}$, *and* a *is a numerical constant.*

Proof. Denote

$$\widehat{f^i}(t) = n^{-1}(2\pi\varepsilon)^{-1/2}[\sum_{j\ne i} \exp\Big(-(t-u_j)^2/2\varepsilon\Big)+1], \quad i = 1,\dots,n.$$

We find that

$$\begin{aligned}\frac{\rho_{32}}{2} &\le y^2\mathbf{E}\ n^{-1}\sum_i r^2(u_i,d)\ \operatorname{ind}(\widehat{f^i}(u_i) \le \delta)\\ &= y^2 n^{-1}\sum_i\ \mathbf{E}r^2(u_i,d)\left[\mathbf{P}(\widehat{f}(t) \le \delta)\right]_{t=u_i}\\ &= y^2\int r^2(t,d)\,\mathbf{P}(\widehat{f}(t)\le\delta)\,f(t,y)\,dt,\end{aligned}$$

where $d = y+\varepsilon$, $\varepsilon > 0$. Here $f(t,y) \le d^{1/2}y^{-1/2}f(t,d)$, and it follows that the right hand side ρ_{32} is not larger than

$$\int y^{3/2}d^{1/2}r^2(t,d)\,\mathbf{P}(\widehat{f}(t)\le\delta)\,f(t,d)\,dt, \quad d = y+\varepsilon, \quad \varepsilon > 0.$$

We rewrite this integral as a sum of two integrals I_1 and I_2 over the regions $\mathfrak{D}_1 = \{t\colon f(t,d) \ge 2\delta\}$ and $\mathfrak{D}_2 = \{t\colon f(t,d) < 2\delta\}$, $\rho_{32} \le I_1 + I_2$. For $t \in \mathfrak{D}_1$, by the Chebyshev inequality, we have

$$\mathbf{P}(\widehat{f}(t) < \delta) \le \mathbf{P}\Big(|\widehat{f}(t) - \mathbf{E}\ \widehat{f}(t)|/\sigma > \delta/\sigma\Big) \le \sigma^2/\delta^2\,,$$

where $\sigma^2 = \operatorname{var}\widehat{f}(t) \le n^{-1}(8\pi^2\varepsilon y)^{-1/2}$. For $\delta = \lambda/\sqrt{2\pi d}$, the ratio $\sigma^2/\delta^2 < n^{-1}(y/\varepsilon)^{1/2}\lambda^{-2}$. By Remark 3, $\int r^2(t,d)\,f(t,d)\,dt \le 1/d$, and it follows that $I_1 \le yn^{-1}(y/\varepsilon)^{1/2}\lambda^{-2}$. Now we estimate I_2. To be more concise denote $e_i = \exp\,(-(t-v_i)^2/2d)$ and let us denote averages over $i = 1,\dots,n$, by angular brackets. For example, $f(t,d) = (2\pi d)^{-1/2}\langle e_i\rangle$. We find that

$$\begin{aligned}I_2 &= \Big(\frac{y}{d}\Big)^{3/2}\int_{\mathfrak{D}_2}\frac{\langle e_i(t-v_i)\rangle^2}{\langle e_i\rangle}\,\mathbf{P}(\widehat{f}(t)<\delta)\,f(t,d)\,dt\\ &\le \Big(\frac{y}{d}\Big)^{3/2}(2\pi d)^{-1/2}\int_{\mathfrak{D}_2}\langle e_i(t-v_i)^2\rangle\ dt.\end{aligned}$$

Let $0 < \alpha < 1$ and $\beta = (1-\alpha)/2$. The product $e_i^{\beta}(t - v_i)^2 \le d/\beta$ for each i. It follows that

$$I_2 \le (2\pi)^{-1/2} y^{3/2} \beta^{-1} d^{-1} \int_{\mathfrak{D}_2} \langle e_i^{\alpha+\beta} \rangle \, dt.$$

Denote $\psi(t) = \pi^{-1} l (l^2 + t^2)^{-1}$, $l > 0$. The integral in the right hand side of the last inequality is not greater than

$$\int_{\mathfrak{D}_2} \sqrt{\langle e_i^{2\alpha} \rangle \, \langle e_i^{2\beta} \rangle} \, dt \le \lambda^{\alpha} \int_{\mathfrak{D}_2} \sqrt{\langle e_i^{2\beta} \rangle} \, dt \ \le \lambda^{\alpha} \Big[\int_{\mathfrak{D}_2} \langle e_i^{2\beta} \rangle \psi(t) \, dt \Big]^{1/2}.$$

The last integral equals

$$\int \langle e_i^{2\beta} \rangle \, \pi l^{-1} (l^2 + t^2) \, dt = \pi^{3/2} \frac{d}{\beta} \Big(2l + \frac{d}{2l\beta} \Big) \le 2\pi^{3/2} \Big(\frac{d}{\beta} \Big)^{3/2}$$

if we choose the quantity l so that $4l^2\beta = d$. We obtain the inequality $I_2 < 2^{3/2} \pi^{1/4} \lambda^{\alpha} (1-\alpha)^{-3/2} y$. This proves Lemma 6.4. □

Theorem 6.1. *There exist functions $\varepsilon = \varepsilon(n, N) < y$ and $\delta = \delta(n, N)$, for example, $\varepsilon = yn^{-18/143}$, and $\delta = y^{-1/2} n^{-40/143}$, such that for any positive integer n and N, the quadratic risk of the estimator* (3) *is*

$$R(\widehat{\mathbf{v}}^{\text{opt}}) \le \big(\sqrt{R_{\text{opt}}} + \sqrt{\rho} \big)^2,$$

where $0 < \rho < ayn^{-1/4}$ and a is a numerical constant.

Proof. We start from Lemmas 2, 3 and 4. It is easy to see that

$$R(\widehat{\mathbf{r}}^{\text{opt}}) = \mathbf{E}\, n^{-1} (\widehat{\mathbf{v}}^{\text{opt}} - \mathbf{v})^2$$
$$\le R_{\text{opt}} + 2\sqrt{R_{\text{opt}}} (\sqrt{\rho_2} + \sqrt{\rho_{31}} + \sqrt{\rho_{32}}) + 5\rho_2 + 2\rho_{31} + 2\rho_{32}.$$

For example, put $\alpha = 0.9$. We note that, by Lemmas 6.3 and 6.4, the first summand in the estimate of the quantity ρ_{32} is not greater than the estimate of ρ_{31} for $\varepsilon \le y$. Let us equate ρ_2, ρ_{31} and the second summand of ρ_{32} (without coefficients). We find that $y\theta^2 = n^{-1}\delta^{-2}\theta^{-3/2} = y^{1.45}\delta^{0.9}$, where $\theta = \varepsilon/y$ as in the above. These equations are satisfied with

$$\theta = yn^{-18/143}, \quad \delta\sqrt{y} = n^{-40/143}. \quad \rho = yn^{-36/143} \le yn^{-1/4}.$$

The proof of Theorem 6.1 is complete. □

Corollary. The estimator (3) with $\varepsilon = \varepsilon(n, N)$ and $\delta = \delta(n, N)$ from Theorem 6.1 is asymptotically dominating over $\mathfrak{K}_1$ as follows: for any c and $\omega > 0$ there exists an integer $n_0 = n_0(c, \omega)$ such that for all n and N satisfying the inequality $n_0 < n < cN$, for any $\widehat{\mathbf{v}} \in \mathfrak{K}_1$, we have

$$R(\widehat{\mathbf{v}}^{\mathrm{opt}}) < \inf_{\widehat{\mathbf{v}} \in \mathfrak{K}_1} R(\widehat{\mathbf{v}}) + \varepsilon.$$

The relative quadratic risk of the estimator $\mathbf{v}^{\mathrm{opt}}$ with the improved unknown estimation function $\varphi^{\mathrm{opt}}(t)$ is $r_{\mathrm{opt}} = R(\varphi^{\mathrm{opt}})/y < 1$. The relative quadratic risk of the estimator (3) is not greater $(\sqrt{r_{\mathrm{opt}}} + a\ n^{-1/4})^2$, where a is a numerical coefficient. In this sense, the estimators $\widehat{\mathbf{v}}^{\mathrm{opt}}$ are asymptotically dominating the class of estimators $\mathfrak{K}_1$ as $n \to \infty$ uniformly with respect to populations and sample size as well.

We note that the assumption of normality can be replaced by the assumption of asymptotic normality for large n and N. In view of the Normal Evaluation Principle of Chapter 4, one can expect that Theorem 6.1 can be extended to a large class of populations.