CHAPTER VII

IMPROVED ESTIMATORS OF HIGH-DIMENSIONAL EXPECTATION VECTORS

Until recently efforts to improve estimators of the expectation value vector were restricted to a special case of shrinkage estimators, that is, estimators with a scalar multiple of the sample mean [23], [25]. The distributions were assumed to be normal or centrally symmetric. In the previous chapter we considered component-wise estimators for vectors with independent components. Now we look for improved estimators of the expectation vectors for dependent variables. We start from an idea to shrink variables in a component-wise manner as in Chapter 6, but for *approximately* independent variables that are produced by passing to the system of coordinates, where the sample covariance matrix is diagonal. Thus the shrinkage coefficients will depend on the sample covariance matrix; we assume that they do not depend on any other variables including sample means.

Define a class $\mathfrak{K}$ of estimators of expectation value vectors $\mathbf{E}\,\mathbf{x} = \mu = (\mu_1, \ldots, \mu_n)$ of the form

$$\widehat{\mu} = \Gamma(C)\bar{\mathbf{x}}, \tag{1}$$

where C is the sample covariance matrix defined in Introduction, and the matrix function $\Gamma(C)$ can be diagonalized together with C with $\Gamma(\lambda)$ as eigenvalues, where λ are corresponding eigenvalues of C; the scalar function $\Gamma : \mathbb{R}^1 \to \mathbb{R}^1$ (denoted by the same letter)

$$\Gamma(u) = \int_{t \geq 0} (1 + ut)^{-1} d\eta(t), \tag{2}$$

has a finite variation on $[0, \infty)$, is continuous except, perhaps, of a finite number of points, and there exist sufficiently many moments

$$\int u^k |d\eta(u)|, \quad k = 1, 2, \ldots .$$

Our problem is to find a function $\Gamma(u)$ minimizing the square losses

$$L_n = L_n(\eta) = (\mu - \widehat{\mu})^2. \tag{3}$$

Limit Quadratic Risk for a Class of Estimators of Expectation Vectors

We use the increasing dimension approach in the limit form as follows.

Consider a sequence of problems

$$\mathfrak{P} = \{(\mathfrak{S}, \mu, N, \mathfrak{X}, \bar{\mathbf{x}}, C, \widehat{\mu})_n\}, \quad n = 1, 2, \ldots$$

(we do not write out the subscripts for the arguments of $\mathfrak{P}$), in which the expectation value vectors $\mu = \mathbf{E}\ \mathbf{x}$ are estimated by samples $\mathfrak{X}$ of size N from populations $\mathfrak{S}$ with sample means $\bar{\mathbf{x}}$ and sample covariance matrices C, and estimators of μ are constructed using an a priori chosen function $\Gamma(u)$.

We restrict the populations with an only requirement that the eight moments of all variables exist. Similiarly to Chapter 2, define

$$M_8 = \max \left[(\mu^2)^4,\ \mathbf{E} \sup_{|\mathbf{e}|=1} (\mathbf{e}^T \overset{o}{\mathbf{x}})^8 \right] > 0,$$

$$\gamma = \sup_{\|\Omega\|<1} \operatorname{var}\ (\overset{o}{\mathbf{x}}{}^T \Omega \overset{o}{\mathbf{x}}) / \sqrt{M_8},$$

where $\overset{o}{\mathbf{x}} = \mathbf{x} - \mu$ is the centered observation vector, $\mathbf{e}$ is a non-random unity vector, and Ω are non-random symmetric matrices of spectral norm not greater 1.

Define the empirical distribution functions

$$F_{0n}(u) = n^{-1} \sum_{i=1}^{n} \operatorname{ind}\ (\lambda_i^0 \le u), \quad G_n(u) = \sum_{i=1}^{n} \mu_i^2 \operatorname{ind}\ (\lambda_i^0 \le u),$$

$$F_n(u) = n^{-1} \sum_{i=1}^{n} \operatorname{ind}\ (\lambda_i \le u),$$

where λ_i^0 are eigenvalues of Σ, λ_i are eigenvalues of C, and μ_i are components of μ in the system of coordinates, in which the matrix Σ is diagonal, $i = 1, \ldots, n$.

To obtain limit relations we restrict $\mathfrak{P}$ with the following conditions:

A. The parameters $0 < M_8 < c_0$ and $\gamma \to 0$ in $\mathfrak{P}$, where c_0 does not depend on n.

B. For each n, all eigenvalues of Σ are located on a segment $[c_1, c_2]$, where $c_1 > 0$ and c_2 do not depend on n.

C. The ratios $n/N \to y$.

D. For $u \geq 0$, the weak convergence holds
$F_{0n}(u) \to F_0(u)$.

E. For $u \geq 0$ almost everywhere, the convergence holds
$G_n(u) \to G(u)$.

Under these conditions the limit exists

$$B \stackrel{\text{def}}{=} \lim_{n\to\infty} \mu^2 = G(c_2).$$

In this chapter we start from the results of the spectral theory of high-dimensional covariance matrices. We consider the resolvent $H = H(z) = (I - zC)^{-1}$ of sample covariance matrices C and use the following corollary of theorems from Chapter 2.

Denote the limits in the square mean by l.i.m. and the convergence in the square mean by the sign $\xrightarrow{2}$.

Let $\mathfrak{G} = \mathfrak{G}(\varepsilon)$ be a region of the complex plane outside some ε-neighbourhood of the axis $z > 0$.

Lemma 7.1. *Under assumptions A–E,*

1^o *The limits exist*

$$h(z) = \operatorname*{l.i.m.}_{n\to\infty} n^{-1}\mathrm{tr}\,(I - zC)^{-1} = \lim_{n\to\infty} n^{-1}\mathrm{tr}\,(I - zs(z)\Sigma)^{-1},$$

where the convergence is uniform in $\mathfrak{G}$;

2^o *the limits exist*

$$b(z) = \operatorname*{l.i.m.}_{n\to\infty} \mu^T(I - zC)^{-1}\mu, \quad k(z) = \operatorname*{l.i.m.}_{n\to\infty} \bar{\mathbf{x}}^T(I - zC)^{-1}\bar{\mathbf{x}},$$

where the convergence is uniform in $\mathfrak{G}$, and

$$k(z) = \begin{cases} b(z) + y(h(z) - 1)/s(z) & \text{if} \quad z \neq 0, \\ B + y\Lambda_1 & \text{if} \quad z = 0, \end{cases}$$

and $\Lambda_1 = \lim_{n\to\infty} n^{-1}$ tr Σ;

3^o *for* $u \geq 0$ *almost everywhere, the limit exists*

$$F(u) = \underset{n\to\infty}{\text{l.i.m.}}\ F_n(u),$$

and $F(u_2) = 1$, *where* $u_2 = c_2(1+\sqrt{y})^2$;

4^o *the equations hold*

$$h(z) = \int (1 - zu)^{-1} dF(u) = \int (1 - zs(z))^{-1} dF_0(u),$$
$$b(z) = \int (1 - zs(z)u)^{-1} dG(u).$$

5^o *the inequality holds* $|h(z) - h(z')| < c|z - z'|^{\zeta}$, *where* $c, \zeta > 0$;

6^o *if* $y < 1$ *and* $|z| \to \infty$, *then* $h(z) \to 0$, $b(z) \to 0$, and $k(z) \to 0$ *in such a way that*

$$h(z) \approx -\Lambda_{-1}(1-y)^{-1} z^{-1}, \quad b(z) \approx -(1-y)^{-1} \int u^{-1} dG_0(u) \cdot z^{-1};$$

if $y > 1$ *and* $|z| \to \infty$,
then $b(z) \to b(\infty)$ *and* $k(z) \to k(\infty)$ *so that*

$$h(z) \approx 1 - y^{-1} - \lambda_0 y^{-1} z^{-1},$$
$$b(z) \approx b(\infty) - \beta \int \frac{u}{(1+\lambda_0 u)^2} dG(u) \cdot z^{-1},$$
$$k(z) - b(z) \approx k(\infty) - b(\infty) - \beta \lambda_0^{-2} z^{-1},$$

where λ_0 *is a root of the equations*

$$\int (1 + \lambda_0 t)^{-1} dF_0(u) = 1 - y^{-1}, \quad \beta = \frac{\lambda_0}{y} \int \frac{u}{(1+\lambda_0 u)^2} dF_0(u),$$
$$b(\infty) = \int (1 + \lambda_0 u)^{-1} dG(u), \qquad k(\infty) = b(\infty) + \frac{1}{\lambda_0}.$$

Lemma 7.2. *Under assumptions A–E uniformly in $z, z' \in \mathfrak{G}$ as $n \to \infty$, the convergence holds*

$$\mu^T H(z)H(z')\mu \xrightarrow{2} \frac{zb(z) - z'b(z')}{z - z'},$$

$$\bar{\mathbf{x}}^T H(z)H(z')\bar{\mathbf{x}} \xrightarrow{2} \frac{zk(z) - z'k(z')}{z - z'}, \tag{4}$$

where the right hand sides are extended by continuity to $z = z'$.

Proof. Let $\varepsilon > 0$ be arbitrarily small. Denote $b_n(z) = \mu^T H(z)\mu$, $k_n(z) = \bar{\mathbf{x}}^T H(z)\bar{\mathbf{x}}$. We have $H(z)H(z') = (zH(z) - z'H(z'))/(z - z')$. By Lemma 7.1, for $z, z' \in \mathfrak{G}$ and $|z - z'| > \varepsilon > 0$ uniformly, there holds the convergence

$$X \stackrel{\text{def}}{=} \mu^T H(z)H(z')\mu = \frac{zb_n(z) - z'b_n(z')}{z - z'} \xrightarrow{2} \frac{zb(z) - z'b(z')}{z - z'},$$

$$Y \stackrel{\text{def}}{=} \bar{\mathbf{x}}^T H(z)H(z')\bar{\bar{\mathbf{x}}} \frac{zk_n(z) - z'k_n(z')}{z - z'} \xrightarrow{2} \frac{zk(z) - z'k(z')}{z - z'}.$$

Suppose $|z - z'| < \varepsilon$. It suffices to prove that X and Y can be written in the form

$$X = \frac{d}{dz}(zb(z)) + \xi_n + \eta(\varepsilon)$$
$$Y = \frac{d}{dz}(zk(z)) + \xi_n + \eta(\varepsilon),$$

where $\xi_n \xrightarrow{2} 0$ as $n \to \infty$ uniformly with respect to z and ε, and $\mathbf{E}\,|\eta(\varepsilon)|^2 \to 0$ as $\varepsilon \to +0$ uniformly in z and n.

Indeed, using the identity

$$H(z)H(z') = \frac{d}{dz}(zH(z)) + (z' - z)CH^2(z)H(z'),$$

we obtain

$$X = \frac{d}{dz}(zb_n(z)) + \xi_n + \eta(\varepsilon),$$

where $\eta(\varepsilon) = (z'-z)\mu^T CH^2(z)H(z')\mu$. Here $\xi_n \xrightarrow{2} 0$ since the second derivatives $b_n''(z)$, and $b''(z)$ exist and are uniformly bounded, and

$$\mathbf{E}\ \left|\frac{d^2}{dz^2} zb_n(z)\right|^2 = 2\mathbf{E}\ \left|\mu^T CH^3(z)\mu\right|^2 = O(1)\mathbf{E}\ \left|\mu^T C\mu\right|^2 = O(1).$$

As $\varepsilon \to +0$, we have

$$\mathbf{E}\ |\eta(\varepsilon)|^2 = O(\varepsilon^2)\mathbf{E}\ \left|\mu^T C\mu\right|^2 = O(\varepsilon^2)$$

uniformly in n. This proves the first statement of our lemma.

Analogously, we rewrite the expression for Y in the form

$$Y = \frac{d}{dz}(zk_n(z)) + \xi_n + \eta(\varepsilon),$$

where $\eta(\varepsilon) = (z' - z)\bar{\mathbf{x}}^T CH(z)H(z')\bar{\mathbf{x}}$. Here $\xi_n \xrightarrow{2} 0$ in view of the convergence $k_n(z) \xrightarrow{2} k(z)$ and the existence and uniform boundedness of the moments

$$\mathbf{E}\ \left|\frac{d^2}{dz^2} zk_n(z)\right|^2 = O(1)\ \mathbf{E}\ |\bar{\mathbf{x}}^T C\bar{\mathbf{x}}|^2 \le O(1)\ \mathbf{E}\ |\bar{\mathbf{x}}^T S\bar{\mathbf{x}}|^2.$$

Indeed, $\mathbf{E}\ |\bar{\mathbf{x}}^T S\bar{\mathbf{x}}|^2 \le 2(\mathbf{E}\ |\mu^T S\mu|^2 + \mathbf{E}\ |\overset{o}{\bar{\mathbf{x}}}{}^T S\dot{\bar{\mathbf{x}}}|^2)$, $\overset{o}{\bar{\mathbf{x}}} = \bar{\mathbf{x}} - \mu$, and

$$\mathbf{E}\ |\overset{o}{\bar{\mathbf{x}}}{}^T S\overset{o}{\bar{\mathbf{x}}}|^2 \le N^{-2}\mathbf{E}\ (\mathrm{tr}^2\ S + 2\ \mathrm{tr}\ S^4) = O(1).$$

Therefore

$$\mathbf{E}\ \left|\eta_n(z)\right|^2 \le O(\varepsilon^2)\mathbf{E}\ |\bar{\mathbf{x}}^T C\bar{\mathbf{x}}|^2 = O(\varepsilon^2).$$

This completes the proof of Lemma 7.2. □

Corollary. Under assumptions A–E as $n \to \infty$,

$$\bar{\mathbf{x}}^T \Gamma^2(C)\bar{\mathbf{x}} \xrightarrow{2} \frac{zk(z) - z'k(z')}{z - z'}.$$

THEOREM 7.1. *Under assumptions A–E we have*

$$R = R(\eta) \overset{\text{def}}{=} \underset{n\to\infty}{\text{l.i.m.}}\ L_n(\eta) =$$
$$= B - 2\int b(-t)d\eta(t) + \iint \frac{tk(-t) - t'k(-t')}{t-t'}\ d\eta(t)d\eta(t'),$$

where the expression in the last integrand is extended by continuity to $t = t'$.

Proof. We have

$$L_n(\eta) = \mu^2 - 2\int \mu^T(I+tC)^{-1}\bar{\mathbf{x}}\ d\eta(t) + \iint \bar{\mathbf{x}}^T(I+tC)^{-1}(I+t'C)^{-1}\bar{\mathbf{x}}\ d\eta(t)d\eta(t'). \qquad (5)$$

We have $\mu^2 \to B$ as $n \to \infty$. Denote $H = (I+tC)^{-1}$, $H' = (I+t'C)^{-1}$. The product $HH' = (tH - t'H')/(t-t')$. In the right hand side of (5), we obtain random values converging to the limits

$$\mu^T H\bar{\mathbf{x}} \xrightarrow{2} b(-t), \quad \bar{\mathbf{x}}^T HH'\bar{\mathbf{x}} \xrightarrow{2} (tk(-t) - t'k(-t'))/(t-t')$$

as $n \to \infty$ uniformly with respect to t, t' by Lemma 7.1 and Lemma 7.2. We conclude that we can perform the limit transition in the integrands in (5). This proves Theorem 7.1. □

Example. Let the matrices $\Gamma(C)$ have a "ridge" form: the function $\eta(v) = 0$ for $v < t$ and $\eta(v) = \alpha > 0$ for $v \geq t$, corresponding to the estimator $\widehat{\mu} = \alpha(I+tC)^{-1}$. In this case

$$R = B - 2\alpha b(-t) + \alpha^2 \frac{d}{dt}(t\cdot k(-t)).$$

Let $t = 0$, $\alpha = 1$, $\widehat{\mu} = \bar{\mathbf{x}}$ (the standard estimator). Then the quadratic risk $R = R^{st} \overset{def}{=} y\Lambda_1$.

Let $t = 0$, $\widehat{\mu} = \alpha\bar{\mathbf{x}}$, where α is a constant. Then we obtain that $R = B(1-\alpha)^2 + \alpha^2 y\Lambda_1$ and the minimum of R equal to $R^{st}B/(B + y\Lambda_1)$ is attained for $\alpha = B/(B+y\Lambda_1)$.

Let $t \neq 0$. The minimum of R equal to $R^{\text{opt}} = B - \alpha^0 b(-t)$ is attained for $\alpha = \alpha^0 = b(-t)/(t\ k(-t))$.

In a special case of the 'ρ-model' (see Chapter 2) of limit spectra of the matrices Σ with of a special choice of identical $\mu_i^2 = \mu^2/n$ for $i = 1, \dots, n,$, we can express the values α^0 and R^{opt} in the form of rational functions of $h(-t)$. Then $b(z) = Bh(z)$. For $\rho = 0$, we obtain $R = R^{st}B(B + y\Lambda_1)^{-1}(1 - y(1-h)^2)$ and the minimum is attained for $h = 1$ with $t = 0$, where $h = h(-t)$.

Corollary. Under assumptions of Theorem 7.1, the equality

$$\int \frac{tk(-t) - t'k(-t')}{t - t'}\, d\eta^0(t') = b(-t), \quad t \geq 0$$

is sufficient for $R(\eta)$ to have a minimum for $\eta(t) = \eta^0(t)$.

Minimization of the Limit Quadratic Risk

To seek for a solution of this equation, we use the analytic properties of functions $h(z)$, $s(z)$, $b(z)$, and $k(z)$. Define

$$\tilde{h}(z) = h(z) - h(\infty), \quad \tilde{b}(z) = b(z) - b(\infty), \quad \tilde{k}(z) = k(z) - k(\infty).$$

LEMMA 7.3. *Suppose conditions A–E hold and $y \neq 1$. Then for any small $\sigma > 0$, we have*

$$\tilde{R}(\Gamma) \overset{\text{def}}{=} B - \frac{1}{\pi i}\int\limits_{\text{Ł}} \frac{\tilde{b}(z)}{z}\Gamma\left(\frac{1}{z}\right) dz + \frac{1}{2\pi i}\int\limits_{\text{Ł}} \frac{\tilde{k}(z)}{z}\Gamma^2\left(\frac{1}{z}\right) dz$$

$$= \begin{cases} R(\Gamma) & \text{if } \ y < 1, \\ R(\Gamma) + 2\Gamma(0)b(\infty) - \Gamma^2(0)k(\infty) & \text{if } \ y > 1, \end{cases} \tag{6}$$

where the contour of the integration Ł $= (\sigma - i\infty, \sigma + i\infty)$.

Proof. The functions $h(z)$, $b(z)$, and $k(z)$ are analytical and have no singularities for Re $z < \sigma$, where $0 < \sigma < u_2^{-1}$. In the half-plane to right of Ł, the functions $\tilde{b}(z)$ and $\tilde{k}(z)$ decrease as $O(|z|^{-1})$ for $|z| \to \infty$ and the function $\Gamma(z^{-1})z^{-1} = O(|z|^{-1})$. Therefore the integration contour Ł can be closed by a semicircle of radius $r = |z - \sigma| \to \infty$. We change the order of integration and use the residue theorem. If follows

$$\frac{1}{2\pi i}\int\limits_{\text{Ł}} \frac{\tilde{b}(z)}{z}\Gamma\left(\frac{1}{z}\right) dz = \int \frac{1}{2\pi i}\oint \tilde{b}(z)(z+t)^{-1} dz d\eta(t)$$

$$= \int \tilde{b}(-t) d\eta(t).$$

The last integral in the right hand side of (6) can be rewritten as

$$\frac{1}{2\pi i}\int\limits_{L}\frac{\tilde{k}(z)}{z}\Gamma^2\left(\frac{1}{z}\right)dz = \iint \frac{1}{2\pi i}\oint \frac{z\tilde{k}(z)}{(z+t)(z+t')}dz\; d\eta(t)d\eta(t').$$

For $t \neq t'$, we calculate two residues at the points $z = -t$ and $z = -t'$ and obtain in the integrand

$$\frac{t\tilde{k}(-t) - t'\tilde{k}(-t')}{t - t'} = \frac{tk(-t) - t'k(-t')}{t - t'} - k(\infty).$$

For $t = t'$, the residue of the second order yields $\frac{d}{dt}(tk(-t))$. We gather summands and obtain the right hand side of (4). Lemma 7.3 is proved. □

Now we consider a special case when the above integrals can be reduced to integrals over a segment. By Lemma 7.1, for any $v > 0$, we have $h(z) \to h(v)$, where $z = v + i\varepsilon$, $\varepsilon \to +0$, and $\operatorname{Im} h(v) > 0$ if and only if $dF(v^{-1}) > 0$. For $b(z)$ to be continuous, an additional assumption is required. We will need the Hölder condition

$$|b(z) - b(z')| < c|z - z'|^{\zeta}, \quad c, \zeta > 0. \tag{7}$$

Remark 1. Condition (7) holds if $0 < y < 1$ and

$$\sup_{u \geq 0}\left|\frac{dG(u)}{dF_0(u)}\right| < a_0,$$

where a_0 is a constant.

Let us prove this assertion. Using the expression for $b(z)$ in statement 4 of Lemma 7.1, we find that $|b(z) - b(z')|$ is not greater than $f(z, z') = |zs(z) - z's(z')|$ with the coefficient

$$\left(\int \frac{u}{|1 - zs(z)u|^2}dG(u)\int \frac{u}{|1 - z's(z')u|^2}dG(u)\right)^{1/2}$$

Note that each of these integrals does not exceed a constant, since we have

$$\int \frac{u}{|1 - zs(z)u|^2}dF_0(u) = \frac{\operatorname{Im}\, h(z)}{\operatorname{Im}\,(zs(z))},$$

and the quantity

$$\begin{aligned}\text{Im } (zs(z)) &= (1-y)\text{ Im } z + y\varepsilon \int |1-zu|^{-2} dF_0(u) \\ &\geq \frac{y\varepsilon}{u_2} \int \frac{u}{|1-zu|^2} dF(u) = \frac{y \text{ Im } h(z)}{u_2},\end{aligned}$$

where $\varepsilon = \text{Im } z \geq 0$. We conclude that the coefficient of $f(z, z')$ in $|b(z) - b(z')|$ is bounded. By Lemma 7.1, $f(z, z')$ satisfies the Hölder condition. The statement of Remark 1 follows. □

Suppose (7) holds. Then the limits exist $b(v) = \lim b(v + i\varepsilon)$ and $k(v) = y(h(v) - 1)/(vs(v)) = \lim k(z)$ as $z = v + i\varepsilon \to v$.

Define the region $\mathfrak{V} = \{v > 0 : \text{ Im } k(v) > 0\}$, and the function

$$\Gamma^0(u) = \text{Im } b(u^{-1})/\text{Im } k(u^{-1}), \quad u^{-1} \in \mathfrak{V}.$$

Let $\Gamma^0(0) = 0$ for $y \leq 1$ and $\Gamma^0(0) = b(\infty)/k(\infty)$ for $y > 1$. Note that $0 \leq \Gamma^0(u) \leq 1$.

THEOREM 7.2. *Suppose conditions A–E and* (7) *hold and* $y \neq 1$. *Then* $R = R(\eta) = R(\Gamma)$ *is such that*

$$\begin{aligned}R = R^{\text{opt}} &+ \frac{1}{\pi} \int\limits_{\mathfrak{V}} \text{Im } k(v) \left(\Gamma\left(\frac{1}{v}\right) - \Gamma^0\left(\frac{1}{v}\right)\right)^2 v^{-1} dv \\ &+ k(\infty)(\Gamma(0) - \Gamma^0(0))^2, \qquad (8)\end{aligned}$$

where

$$R^{\text{opt}} = B - \frac{1}{\pi} \int\limits_{\mathfrak{V}} \frac{(\text{Im } b(v))^2}{\text{Im } k(v)} v^{-1} dv - \Gamma^0(0) b(\infty). \qquad (9)$$

Proof. Given $\text{Im } z > 0$, we contract the contour Ł to the beam $z \geq \sigma > 0$ using analytical properties of functions in (6). The function

$$\alpha(z) = \frac{1}{z} \Gamma\left(\frac{1}{z}\right) = \int\limits_{t \geq 0} (z+t)^{-1} d\eta(t)$$

has a bounded derivative for $\text{Re } z > \sigma^{-1}$ and the expressions in the integrands in (6) satisfy the Hölder inequality. The contribution of large $|z|$ can be made arbitrarily small. Under the contraction of Ł to real points outside of $\mathfrak{V}$, contributions of integrals of $b(z)$ and $k(z)$ along oppositely directed parallel beams on both sides from the axis of abscissae mutually cancel. The contributions of real parts of $b(z)$ and $k(z)$ mutually cancel, while contributions of imaginary parts double. We obtain

$$\begin{aligned} R(\Gamma) =& B - \frac{2}{\pi} \int\limits_{\mathfrak{V}} \operatorname{Im}\, b(v)\Gamma\left(\frac{1}{v}\right) v^{-1} dv \\ &+ \int\limits_{\mathfrak{V}} \operatorname{Im}\, k(v)\Gamma^2\left(\frac{1}{v}\right) v^{-1} dv - 2\Gamma(0)b(\infty) + \Gamma^2(0)k(\infty). \end{aligned} \tag{10}$$

The right hand side of (8) coincides with the right hand of (10). This is the proof of Theorem 7.2. □

Denote $\mathfrak{U}_0 = \{u : u = 0 \text{ or } \ u^{-1} \in \mathfrak{V}\}$.

Remark 2. Suppose conditions A–E hold, $y \neq 1$, and there exists a function of finite variation $\eta^0(t)$ such that

$$\int\limits_{t \geq 0} (1 + ut)^{-1} d\eta^0(t) = \Gamma^0(u) \quad for \quad u \in \mathfrak{U}_0.$$

Then $\underset{n\to\infty}{\text{l.i.m.}}\ (\mu - \Gamma^0(C)\bar{\mathbf{x}})^2 = R^{\text{opt}}$.

Example. In a special case, let $\mu_i^2 = \mu^2/n, \quad i = 1, \dots, n,$ for all n in the system of coordinates where Σ is diagonal. By Lemma 7.1, we have $b(\infty) = 0$ if $y < 1$ and $b(\infty) = B(1 - y^{-1})$ if $y > 0$. The functionals $\tilde{R}(\Gamma)$ and $R(\Gamma)$ can be expressed in the form of integrals over the limit spectrum of matrices C. We find that

$$R(\Gamma) = B - 2B \int \Gamma(u) dF(u) + \int q(u)\Gamma^2(u) dF(u),$$

where

$$q(u) = \begin{cases} B + yu|s(u^{-1})|^{-2} & \text{if } u^{-1} \in \mathfrak{V}, \\ 0 & \text{if } y < 1 \ \text{ and } u = 0, \\ B + (1 - y^{-1})^{-1}\lambda_0^{-1} & \text{if } y > 1 \ \text{ and } u = 0. \end{cases}$$

Assume that the function $F_0(u)$ has a special form defined by the 'ρ-model' of limit spectrum of the matrices Σ. In this case, the function $h(-t)$ equals

$$2\left(1+\rho+\kappa(1-y)t+\sqrt{(1+\rho+\kappa(1-y)t)^2-4\rho+4\kappa yt)}\right),$$

where $\rho<1$ and $\kappa=\sigma^2(1-\rho^2)$ are the model parameters. We find that $|s(v)|^2=(\rho+y(1-\rho))(\rho+\kappa yv)^{-1}$ and the function

$$\Gamma^0(u)=\frac{B}{q(u)}=\frac{\alpha^0}{1+ut^0},$$

where

$$\alpha^0=\frac{B[\rho+y(1-\rho)]}{B[\rho+y(1-\rho)]+\kappa y^2},\quad t^0=\frac{y\rho}{B[\rho+y(1-\rho)]+\kappa y^2}.$$

Let $\eta^0(t')$ be a step-wise function with a jump α^0 at the point $t'=t$. We calculate $R(\Gamma^0)$ passing back to the contour Ł and calculating the residue at the point $z=-t$. It follows that $R^{\text{opt}}=R(\Gamma^0)=B(1-\alpha^0h(-t^0))$. As $y\to 0$, we have $\alpha^0\to 1$, $t^0\to 0$, and $R^0\to 0$, thus showing the advantage of the standard estimator when the dimension increases slower than sample size. If $\Sigma=\sigma^2I$ for all n, then the values $\rho=0$, $\alpha^0=B/(B+y\Lambda_1)$, $\Lambda_1=\sigma^2$, $t^0=0$, $\Gamma^0(C)=\alpha^0I$, and $R^0=\alpha R^{st}$, where $R^{st}=y\Lambda_1$. The corresponding optimum estimator has a shrinkage form $\widehat{\mu}=\alpha^0\bar{\mathbf{x}}$. As $y\to 1$, the values $\alpha^0\to B/(B+\kappa)$, $t^0\to\rho/(B+\kappa)$, $\Gamma^0(C)\to B[(B+\kappa)I+\rho C]^{-1}$, and

$$\frac{R^{\text{opt}}}{R^{st}}\to\theta\left(1-\rho+\frac{\rho\theta}{1+\sqrt{1+\Lambda_1\rho(1-\rho)^2/(B+\kappa).}}\right),$$

where $\theta=B/(B+\kappa)$. The maximum effect of the estimator $\widehat{\mu}=\Gamma^0(C)\bar{\mathbf{x}}$ is achieved for $B\ll\kappa$ that corresponds to the case of small μ^2 when $\mu^2\ll(n^{-1}\text{tr}\,\Sigma^{-1})^{-1}$, when components of μ are much less in absolute value than the standard deviation of components of the sample mean vector, or for a wide spectrum of matrices Σ. If $y\to\infty$, then $R^{st}\to\infty$, whereas $\alpha^0\to 0$ and $R^0\to B<\infty$, whereas $R=y\to\infty$ for the standard estimator.

Statistics to Approximate the Limit Risk Function

Now we pass to the construction of estimators for the involved limit characteristics. Denote

$$h_n(z) = n^{-1}\text{tr } (I - zC)^{-1}, \quad b_n(z) = \mu^T (I - zC)^{-1}\mu,$$

$$s_n(z) = 1 + nN^{-1}(h_n(z) - 1), \quad k_n(z) = b_n(z) + nN^{-1}\frac{h_n(z) - 1}{zs_n(z)}.$$

From Lemma 7.1, it follows that the convergence holds

$$h_n(z) \xrightarrow{2} h(z), \quad b_n(z) \xrightarrow{2} b(z), \quad s_n(z) \xrightarrow{2} s(z),$$
$$n^{-1}\text{tr } C \xrightarrow{2} \Lambda_1, \quad n^{-1}\text{tr } C^2 \xrightarrow{2} \Lambda_2 + y\Lambda_1^2,$$
$$\lim_{T\to\infty} \underset{n\to\infty}{\text{l.i.m.}}\ Ts_n(-T) = \lambda_0, \quad \lim_{T\to\infty} \underset{n\to\infty}{\text{l.i.m.}}\ k_n(-T) = k(\infty).$$

The asymptotically extremal estimator $\widehat{\mu} = \Gamma^0(C)\bar{\mathbf{x}}$ involves the function Im $b(u^{-1})$/Im $k(u^{-1})$, which should be estimated by observations. But the natural estimator

$$\Gamma_n^0(u) = \text{Im } b_n(u^{-1})/\text{Im } k_n(u^{-1})$$

is singular for $u > 0$ and may not approach $\Gamma^0(u)$ as $n \to \infty$. We introduce a smoothing by considering $b_n(z)$ and $k_n(z)$ for complex z with Im $(z) > 0$. In applications, the character of smoothing may be essential. To reach a uniform smoothing, it is convenient to pass to functions of the inverse arguments and deal with functions

$$a_n(z) = z^{-1}b_n(z^{-1}) = \mu^T(zI - C)\mu,$$
$$g_n(z) = z^{-1}h_n(z^{-1}) = n^{-1}\text{tr } (zI - C)^{-1},$$
$$l_n(z) = z^{-1}k_n(z^{-1}) = \bar{\mathbf{x}}^T(zI - C)^{-1}\bar{\mathbf{x}}.$$

Remark 3. Under assumptions A–E, the functions $g_n(z)$, $a_n(z)$, and $l_n(z)$ converge in the square mean uniformly with respect to $z \in \mathfrak{G}$ to the limits $g(z)$, $a(z)$, $l(z)$, respectively, such that

$$g(z) = \int (z - s(z^{-1})u)^{-1}dF_0(u), \quad a(z) = \int (z - s(z^{-1})u)^{-1}dG(u),$$
$$l(z) = a(z) + y(zg(z) - 1)/s(z^{-1}). \tag{11}$$

Remark 4. Under assumptions A–E for $y \neq 1$, the functions $g(z)$, $a(z)$, and $l(z)$ are regular with singularities only at the point $z = 0$ and on the segment $[0, u_2]$. The functions $\tilde{a}(z) = a(z) - b(\infty)/z$ and $\tilde{l}(z) = l(z) - k(\infty)/z$ are bounded. As $z \to u > u_2$, we have $\text{Im } g(z) \to 0$, $\text{Im } a(z) \to 0$, and $\text{Im } l(z) \to 0$.

Now we express (6) in terms of $g(z)$, $s(z)$, and $l(z)$.

LEMMA 7.4. *If conditions A–E hold and $y \neq 1$, then, as $\varepsilon \to +0$, the function $\tilde{R}(\Gamma)$ defined by* (6) *equals*

$$B - \frac{2}{\pi} \int\limits_0^\infty \text{Im } \tilde{a}(u - i\varepsilon)\Gamma(u) du + \frac{1}{\pi} \int\limits_0^\infty \text{Im } \tilde{l}(u - i\varepsilon)\Gamma^2(u) du + O(\varepsilon). \quad (12)$$

Proof. Functions in the integrands in (6) are regular and have no singularities for $\text{Re } z \geq \sigma > 0$ outside the beam $z \geq \sigma$. As $|z| \to \infty$, there exists a real $T > 0$ such that $\tilde{b}(z)$ has no singularities also for $|z| > T$. Let as deform the contour $(\sigma - i\infty, \sigma + i\infty)$ in the integrals (6) into a closed contour Ł$_1$ surrounding an ε-neighbourhood of the segment $[\sigma, T]$. Substitute $w = z^{-1}$. We find that

$$\tilde{R}(\Gamma) = B - \frac{1}{\pi i} \int\limits_{Ł_2} \tilde{a}(w)\Gamma(w) dw + \frac{1}{2\pi i} \int\limits_{Ł_2} \tilde{l}(w)\Gamma^2(w) dw,$$

where Ł$_2$ is surrounding the segment $[w_0, T]$, where $w_0 = T^{-1}$ and $t = \sigma^{-1}$. If $\text{Re } w \geq 0$, then the analytical function $\Gamma(w)$ is bounded by the inequality $|\Gamma(w)| \leq 1$, and $\tilde{a}(w)$, $b(z)$ and $\tilde{l}(w)$ tend to a constant as $w \to u$, where $u = 1/\text{Re } z > 0$. Since the functions $\tilde{a}(w)$ and $\tilde{l}(w)$ are analytical we can deform the contour Ł$_2$ into the contour $\tilde{Ł}_2 = (0 - i\varepsilon, 0 + i\varepsilon, T + i\varepsilon, T - i\varepsilon, 0 - i\varepsilon)$, where $T > T_0 > 0$ is sufficiently large. Contributions of integrals along vertical segments of length 2ε are $O(\varepsilon)$ as $\varepsilon \to +0$. Real parts of the integrands over the segments $[i\varepsilon, \tau + i\varepsilon]$ and $[\tau - i\varepsilon, -i\varepsilon]$ cancel, whilst the imaginary ones double. We obtain

$$\tilde{R}(\Gamma) = B - \frac{2}{\pi} \int\limits_0^T \text{Im } [\tilde{a}(w)\Gamma(w)] du + \frac{1}{\pi} \int\limits_0^T \text{Im } [\tilde{l}(w)\Gamma^2(w)] du + O(\varepsilon),$$

where $w = u - i\varepsilon$. Substitute

$$\Gamma(u - i\varepsilon) = \Gamma(u) + i\varepsilon \int \frac{t}{1+ut} \; \frac{1}{1+ut-i\varepsilon t} \, d\eta(t).$$

Comparing with (12), we see that it is left to prove that the difference $\Gamma(u-i\varepsilon) - \Gamma(u)$ gives a contribution $O(\varepsilon)$ into $\tilde{R}(\Gamma)$. Consider the integral

$$\int\limits_{Ł_3} \tilde{a}(w) \left(\int \frac{1}{1+wt} \; \frac{t}{1+(w+i\varepsilon)t} \right) dw, \tag{13}$$

where the integration contour is $Ł_3 = (0 - i\varepsilon, \infty - i\varepsilon)$. If Im $w < \varepsilon$, then the integrand has no singularities and is $O(|w|^{-2})$ as $|w| \to \infty$. It means that we can replace the contour $Ł_3$ by the contour $Ł_4 = (0 - i\varepsilon, 0 - i\infty)$. The function $\tilde{a}(w)$ is uniformly bounded on $Ł_4$, and it follows that the integral (13) is uniformly bounded. Analogously, the integral with $\tilde{l}(w)$ is also bounded. It follows that we can replace $\Gamma(u-i\varepsilon)$ in $\tilde{R}(\Gamma)$ by $\Gamma(u)$ with the accuracy to $O(\varepsilon)$. We have proved the statement of Lemma 7.4. □

Statistics to Approximate the Extremal Limit Solution

Let us construct an estimator of the extremal limit function $\Gamma^0(u)$. Let $\varepsilon > 0$. Denote

$$\Gamma^0_\varepsilon(u) = \begin{cases} \text{Im } a(u-i\varepsilon)/\text{Im } l(u-i\varepsilon) \le 1, & \text{if } u \ge 0, \\ 0, & \text{if } u < 0; \end{cases}$$

$$R^{\text{opt}}_\varepsilon = B - \frac{1}{\pi} \int\limits_0^\infty \frac{(\text{Im } a(w))^2}{\text{Im } l(w)} \, du - d,$$

where $w = u - i\varepsilon$, $d = 0$ if $y < 1$ and $d = b^2(\infty)/k(\infty)$ if $y > 1$. From (12), we obtain that

$$\begin{aligned} R(\Gamma) = & R^{\text{opt}}_\varepsilon + \frac{1}{\pi} \int\limits_0^\infty \text{Im } l(w) \; (\Gamma(u) - \Gamma^0_\varepsilon(u))^2 du \\ & + k(\infty)(\Gamma(0) - \Gamma^0_\varepsilon(0))^2 + O(\varepsilon). \end{aligned} \tag{14}$$

Obviously, the best estimator among estimators $\widehat{\mu} = \Gamma(C)\bar{\mathbf{x}}$ is such an estimator that approximates better $\Gamma^0_\varepsilon(u)$ for $u \geq 0$ with accuracy to $O(\varepsilon)$.

We consider the smoothed estimator $\tilde{\mu} = \tilde{\Gamma}^0_\varepsilon(C)\bar{\mathbf{x}}$ defined by the scalar function

$$\tilde{\Gamma}^0_\varepsilon(u) = \int\limits_{-\infty}^{\infty} \Gamma^0_\varepsilon(u')\frac{\varepsilon}{\pi} \; \frac{1}{(u-u')^2+\varepsilon^2} \; du'.$$

LEMMA 7.5. *If conditions A–E hold and $y \neq 1$, then*

$$(\mu - \tilde{\Gamma}^0_\varepsilon(C)\bar{\mathbf{x}})^2 < R^{\mathrm{opt}}_\varepsilon + O(\varepsilon) + \xi_n(\varepsilon), \tag{15}$$

where $\mathbf{E}\ \xi_n^2(\varepsilon) \to 0$ *as* $n \to \infty$ *for fixed* $\varepsilon > 0$, *and* $O(\varepsilon)$ *does not depend on* n.

Proof. We pass to the coordinate system where the matrix C is diagonal; let μ_i and $\bar{x}_i$ be components of μ and $\bar{\mathbf{x}}$ therein. We find that

$$(\mu - \tilde{\Gamma}^0_\varepsilon(C)\bar{\mathbf{x}})^2 = \mu^2 - 2\sum_i \mu_i^2\ \tilde{\Gamma}^0_\varepsilon(\lambda_i) + \sum_i \bar{x}_i^2\ \tilde{\Gamma}^{02}_\varepsilon(\lambda_i) - 2\zeta_n, \tag{16}$$

where $\lambda_j = \lambda_j(C)$, $\zeta_n = \mu^T\tilde{\Gamma}^0_\varepsilon(C)(\bar{\mathbf{x}} - \mu)$, and $\mathbf{E}\ \zeta_n^2 = O(N^{-1})$. Note that

$$\sum_j \mu_j^2\tilde{\Gamma}^0_\varepsilon(\lambda_j) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} \mathrm{Im}\ a_n(u - i\varepsilon)\ \Gamma^0_\varepsilon(u)du, \tag{17}$$

where $\lambda_j = \lambda_j(C)$. For a fixed $\varepsilon > 0$, $a_n(w) \to a(w)$ as $n \to \infty$ uniformly on $[0, T]$, where $T = 1/\varepsilon$. The contribution of $u \in [0, T]$ to (17) is not larger than

$$\sum_j \mu_j^2 \left(1 - \frac{1}{\pi}\ \arctan\frac{T-\lambda_j}{\varepsilon} - \frac{1}{\pi}\ \arctan\ (\frac{\lambda_j}{\varepsilon})\right)$$

$$\leq \sum_{\lambda_j > T/2} \mu_j^2 + \frac{\varepsilon^2\mu^2}{2\pi}\ \leq \frac{2}{T}\sum_j \mu_j^2\lambda_j + \frac{\varepsilon^2\mu^2}{2\pi} = \frac{2}{T}\ \mu^T C\mu + \frac{\varepsilon^2\mu^2}{2\pi}. \tag{18}$$

From Lemma 7.1 it follows that $\mathbf{E}\,(\mu^T C\mu)^2$ is bounded and the right hand side of (18) can be expressed in the form $O(\varepsilon)+\xi_n(\varepsilon)$, where $O(\varepsilon)$ is independent on n and $\mathbf{E}\,\xi_n^2(\varepsilon)\to 0$ as $n\to\infty$ for fixed $\varepsilon>0$. Thus the second term of the right hand side of (16) equals

$$-\frac{2}{\pi}\int_0^\infty \mathrm{Im}\; a(u-i\varepsilon)\;\Gamma_\varepsilon^0(u)du+O(\varepsilon)+\xi_n(\varepsilon).$$

We notice that the third term of (16) is

$$\sum_j \bar{\mathbf{x}}_j^2\tilde{\Gamma}_\varepsilon^{02}(\lambda_j)\le\sum_j\bar{\mathbf{x}}_j^2\int_{-\infty}^{\infty}\Gamma_\varepsilon^{02}(u)\frac{\varepsilon/\pi}{(u-\lambda_j)^2+\varepsilon^2}du$$
$$=\frac{1}{\pi}\int_{-\infty}^{+\infty}\Gamma_\varepsilon^{02}(u)\mathrm{Im}\;l_n(w)du, \tag{19}$$

where the second superscript 2 denotes the square, $w=u-i\varepsilon$, $\varepsilon>0$, and $0\le\Gamma_\varepsilon^0(u)\le 1$. Note that $l_n(w)\xrightarrow{2} l(w)$ uniformly for $u\in[0,T]$, and the contribution of $u\,\bar{\in}\,[0,T]$ is not greater than $2T^{-1}\bar{\mathbf{x}}^T C\bar{\mathbf{x}}+\varepsilon^2\bar{\mathbf{x}}^2/(2\pi)$. But we have $\mathbf{E}\;\bar{\mathbf{x}}^2=O(1)$ and $\mathbf{E}\;(\bar{\mathbf{x}}^T C\bar{\mathbf{x}})^2=O(1)$. It follows that the third term of the right hand side of (16) is not greater than

$$\frac{1}{\pi}\int_0^T\Gamma_\varepsilon^{02}(u)\mathrm{Im}\;l(u-i\varepsilon)\;du+O(\varepsilon^2)+\xi_n(\varepsilon),$$

where $O(\varepsilon)$ is finite as $\varepsilon\to+0$ and $\xi_n(\varepsilon)\xrightarrow{2}0$ as $n\to\infty$ for any $\varepsilon>0$. We substitute $\mu^2=B+o(1)$, $\Gamma_\varepsilon^0(u)=\mathrm{Im}\;a(w)/\mathrm{Im}\;l(w)$, where $w=u-i\varepsilon$. Gathering summands, we obtain

$$(\mu-\tilde{\Gamma}_\varepsilon^0(C)\bar{\mathbf{x}})^2<B-\frac{1}{\pi}\int_0^T\frac{(\mathrm{Im}\;a(w))^2}{\mathrm{Im}\;l(w)}du+O(\varepsilon)+\xi_n(\varepsilon), \tag{20}$$

where $w=u-i\varepsilon$. We note that $\mathrm{Im}\;a(w)=O(\varepsilon)\;|w|^{-2}$ as $|w|\to\infty$, and, consequently, the integral in (20) from 0 to T can be replaced by the integral from 0 to infinity with the accuracy to $O(\varepsilon)$. The statement of Lemma 7.5 follows. □

Now, we consider the statistics

$$\Gamma^0_{n\varepsilon}(u) = \max\left(0,\, 1 - nN^{-1}\frac{\operatorname{Im}\,(wg_n(w))}{|s_n(w^{-1})|^2}\right), \quad \text{where } w = u - i\varepsilon,$$

$$\text{and}\quad \tilde{\Gamma}^0_{n\varepsilon}(u) = \int_{-\infty}^{\infty} \Gamma^0_{n\varepsilon}(u')\frac{\varepsilon/\pi}{[(u-u')^2+\varepsilon^2]}\,du'.$$

THEOREM 7.3. *Suppose conditions A–E hold and $y \neq 1$. Then*

1^0 *for a fixed $\varepsilon > 0$ as $n \to \infty$, we have $\tilde{\Gamma}^0_{n\varepsilon}(u) \xrightarrow{2} \tilde{\Gamma}^0_{\varepsilon}(u)$ uniformly on any segment;*

2^0 *we have*

$$(\mu - \tilde{\Gamma}^0_{n\varepsilon}(C)\bar{\mathbf{x}})^2 < \inf_{\Gamma}(\mu - \Gamma(C)\bar{\mathbf{x}})^2 + O(\varepsilon) + \xi_n(\varepsilon), \qquad (21)$$

where $\Gamma = \Gamma(\cdot)$ are from the class $\mathfrak{K}$, the quantity $O(\varepsilon)$ does not depend on n, and $\mathbf{E}\ \xi_n^2(\varepsilon) \to 0$ as $n \to \infty$ for any $\varepsilon > 0$.

Proof. For fixed $\varepsilon > 0$, we have the uniform convergence $\tilde{\Gamma}^0_{n\varepsilon}(u) \xrightarrow{2} \tilde{\Gamma}^0_{\varepsilon}(u)$ on any segment by definition of these functions and Lemma 7.1. Denote

$$\rho_n = (\mu - \tilde{\Gamma}^0_{n\varepsilon}(C)\bar{\mathbf{x}})^2 - (\mu - \tilde{\Gamma}^0_{\varepsilon}(C)\bar{\mathbf{x}})^2, \quad \Delta(C) = \tilde{\Gamma}^0_{n\varepsilon}(C) - \tilde{\Gamma}^0_{\varepsilon}(C),$$

Let us prove that $\lim\limits_{\varepsilon\to+0}\ \lim\limits_{n\to\infty} \mathbf{E}\ \rho_n^2 = 0$. It is suffices to show that $\mathbf{E}\ (\mu^T\Delta\bar{\mathbf{x}})^2 \xrightarrow{2} 0$ and $\mathbf{E}\ (\bar{\mathbf{x}}^T\Delta\bar{\mathbf{x}})^2 \xrightarrow{2} 0$. We single out a contribution of eigenvalues λ_i of C not exceeding T for some $T > 0$: Let $\Delta(u) = \Delta_1(u) + \Delta_2$ (u), where $\Delta_2(u) = \Delta(u)$ for $|u| > T$ and $\Delta_2(u) = 0$ for $|u| \leq T$. Here the scalar argument u stands for eigenvalues of C. By virtue of the first theorem statement, $\Delta_1(u) \xrightarrow{2} 0$ as $n \to \infty$ uniformly on the segment $[0, T]$. The contribution of $|u| > T$ to $\mathbf{E}\ (\mu^T\Delta(C)\bar{\mathbf{x}})^2$ is not greater than

$$\sum_i \mu_j^2 \mathbf{E}\ \mathbf{x}_j^2\ \text{ind}\ (\lambda_j > T) \leq T^{-1}\mathbf{E}\ (\bar{\mathbf{x}}^T C\bar{\mathbf{x}}) = O(T^{-1}).$$

Let $T = 1/\varepsilon$. Then $\lim_{n\to\infty} \mathbf{E}\ \rho_n^2 = O(\varepsilon)$. In view of Lemma 7.5,

$$(\mu - \tilde{\Gamma}^0_{n\varepsilon}(C)\bar{\mathbf{x}})^2 < R^0_\varepsilon + O(\varepsilon) + \xi_n(\varepsilon),$$

where the estimate $O(\varepsilon)$ is uniform in n and $\mathbf{E}\ \xi_n^2 \to 0$ as $n \to \infty$ for fixed $\varepsilon > 0$. It follows that $R^0_\varepsilon \le R(\Gamma) + O(\varepsilon)$. This completes the proof of Theorem 7.3. □

Denote $\widehat{\mu}^0_\varepsilon = \tilde{\Gamma}^0_{n\varepsilon}(C)\bar{\mathbf{x}}$.

We can conclude that in the sequence of problems $\{\mathfrak{P}_n\}$ of estimation of n-dimensional parameters $\mu = \mathbf{E}\ \mathbf{x}$ for populations restricted by conditions A–E, the family of estimators $\{\widehat{\mu}^0_\varepsilon\}$ is asymptotically ε-dominating over the class of estimators $\widehat{\mu} \in \mathfrak{K}$ of μ as follows: for any $\varepsilon > 0$ and $\delta > 0$ there exists an n_0 such that for any $n > n_0$ for any μ for any estimator $\Gamma(C)\bar{\mathbf{x}}$, the inequality

$$(\mu - \widehat{\mu}^0_\varepsilon)^2 < (\mu - \widehat{\mu})^2 + \varepsilon \tag{22}$$

holds with the probability $1 - \delta$.

In the above, it was shown that the estimation of a large number of parameters over samples of limited size produces a bias proportional to n/N. The effect of a decrease of the quadratic risk arising from using of improved estimators is of the same order. Under conditions A–E, the estimator $\widehat{\mu}^0_\varepsilon$ is asymptotically supereffective, provides quadratic losses asymptotically not exceeding $R^{\text{opt}} \le y$ and proves to be ε-unimprovable in the limit.