THEORY OF STOCHASTIC CANONICAL EQUATIONS

VOLUME 1

Vyacheslav L.Girko

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Reviewer No. 021527 V.I.Serdobolskii (Moscow)

This monograph by V.Girko in two volumes summarizes 25 years long author investigations of increasing random matrices. Results of these investigations were published sequentially in a series of his monographs "Random Matrices" (Kiev, 1975, in Russian), "Theory of Random Determinants" (Kiev, 1980, in Russian, translated version: Kluwer Academic Publishers, 1990), "Multidimensional Statistical Analysis" (Kiev, 1988, in Russian), "Spectral Theory of Random Matrices" (Moscow, 1988, in Russian), "Statistical Analysis of Observations of Increasing Dimension" (Kluwer Academic Publishers, 1995), "Theory of Linear Algebraic Equations with Random Coefficients" (N.Y., 1996), "An Introduction to Statistical Analysis of Random Arrays" (VSP, 1998), and in numerous papers. The reader can find latest publications in the journal "Random Operators and Stochastic Equations".

In all these publications V.Girko investigates spectral properties of different random matrices $n \times n$ as $n \to \infty$. The author developed an extremely complicated functional technique necessary and appropriate to describe objects of such high complexity. Mathematical community met his labor-consuming constructions with a sort of reluctance, partly because publications by V.Girko were characterized by a hasty manner with not enough attention to readers and their sincere efforts to understand details. The reviewed monograph profitably differs from the previous ones. It is written as a handbook and presents a systematic review of the author's results in a unique manner and unique notations along with an elaborated system of derivations. Each chapter is a completed investigation and can be understood separately. Still, results of V.Girko's investigations in the monograph are so numerous and multifarious that (even apart from derivations) they cannot be in any way be accurately and completely cited in this short review. We present only a brief general discussion of the contents of the book.

The object of the author's studies are spectral properties of different random matrices $\Xi = \{\xi_{ij}\}$ (here and in the following, $i, j = 1, \dots, n$). The term "stochastic canonic equation" (SCE) has a specifically author's meaning and denotes a relation that holds between limit spectral functions of random matrices Ξ as $n \to \infty$ and of underlying non-random matrices under weak restrictions of a general kind. A more appropriate title of the book would be "Asymptotical theory of spectral equations".

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The first volume contains 30 chapters in which 30 different SCE (called $K_1 - K_{30}$) are derived.

In each case, the convergence of the "counting functions"

$$\mu_n(x) = \frac{1}{n} \sum_i \text{ind } (\lambda_i \leq x),$$

where λ_i are eigenvalues of $\Xi = \{\xi_{ij}\}$, is established. The author proves that $\mu_n(x)$ weakly converge to some non-random distribution functions $F_n(x)$ whose Stiltjes transforms

$$c(z) = \int (x - z)^{-1} dF_n(x), \qquad \text{Im } z \neq 0,$$

can be determined from solutions of CSE of different kind.

It is assumed that the expectation matrix $A = \mathbf{E}\Xi = \{a_{ij}\}$ exists and entries a_{ij} are uniformly bounded by some requirements.

Different conditions are considered restricting values of ξ_{ij} :

1, Variances $\sigma_{ij} = \text{var } \xi_{ij}$ exist for each i, j and are uniformly bounded.

2. Generalized Lindeberg-type conditions hold, for example, for each $\varepsilon > 0$

$$\lim_{\tau \to \infty} \max_i \sum_j \mathbf{E} \ (\xi_{ij} - a_{ij})^2 \text{ind } [(\xi_{ij} - a_{ij})^2 > \tau] \to 0$$

3. For any $\varepsilon > 0$, i, and j, $\mathbf{P}[|\xi_{ij} - \alpha_{ij}| > \varepsilon] \to 0$, where α_{ij} are some constants. This condition is called the "ACE condition" (asymptotically constant entries).

4. The limit distribution $N(x) = N(x, u, v)$ exists and the function $K(x) = K(x, u, v)$ exists such that as $n \to \infty$ uniformly

$$n \left(1 - \mathbf{P}[(\xi_{ij} - a_{ij})^2 < x]\right) \to N(x, u, v),$$
$$n \int_0^x y(1 + y)^{-1} d\mathbf{P}[(\xi_{ij} - a_{ij})^2 < y] \to K(x, u, v),$$

where u and v are continuous variables presenting ratios i/n and j/n, and $N(x, u, v)$ and $K(x, u, v)$ are continuous in u and v.

The major part of the book is devoted to the investigation of symmetric matrices Ξ with independent entries (in the upper triangle) and of and Gram matrices $A + \Xi\ \Xi^T$, where A are symmetrical constant matrices, and Ξ are arbitrary rectangular matrices $n \times m$ with independent entries.

For symmetrical matrices Ξ under conditions 1 and 2 with $\sigma_{ij} = \sigma$ for all i, j, the SCE is found of the form

$$c(z) = n^{-1} \text{tr } [A - zI - \sigma c(z) I]^{-1}, \qquad \text{Im } z \neq 0$$

(here and in the following, I means identity matrix). From this equation, the well-known Wigner spectrum semi-circle law (1957) is deduced for $A = I$ and the "Cubic law" for two different eigenvalues of A. In case of different σ_{ij}, more complicated CSE are obtained.

For Gram matrices $\Xi\ \Xi^T$ with var $\xi_{ij} = n^{-1}\sigma_j, \quad i = 1, \ldots, n$, the equation is found

$$c(z) = n^{-1}\mathrm{tr}\ \left[A - zI + n^{-1}\sum \frac{\sigma_i}{1 + \gamma\sigma_i c(u)} I\right]^{-1}$$

(first obtained by F.Berezin, 1973). In case of σ_{ij} depending on both subscripts, the author obtained more complicated SCE.

For standard sample covariance matrices under the Kolmogorov condition $m/n \to \gamma$, where m is the observation dimension and n is sample size, the author derives the SCE

$$c(z) = m^{-1}\sum_i [(1 - \gamma + z\gamma c(z)\lambda_i - z]^{-1},$$

where λ_i are eigenvectors of true covariance matrices. (first obtained by V.Serdobolskii, 1983). For non-standard sample covariance matrices with known mean vectors, the same equation was derived earlier (L.Pastur, 1973). In the monograph, more general SCE are found for sums of covariance matrices and matrices of constats.

Assuming the ACE condition and condition 4, the SCE is obtained

$$c(z) = \mathbf{P}[(1 + t\xi_\alpha)^{-1} < x],$$

where $z = it$, $t > 0$ is real, and ξ_α are random functionals whose Stiltjes transform are some functions depending on $K(x, u, v)$. These functions are obtained explicitly for symmetrical matrices and for Gram matrices.

Also the author investigates non-Hermitian matrices Ξ of two sorts: matrices with independent pairs of symmetrical entries ξ_{ij} and ξ_{ji} under conditions 1 and 2 and matrices under the ACE condition. He starts from Hermitian matrices $(\tau I - \Xi)(\tau I - \Xi)^T + \alpha I$ ("V-transform") and obtains a number of SCE for these matrices and for Ξ.

In the monograph, V.Girko obtains not only scalar asymptotic equations for normed trace of resolvents and spectral distribution function, but also for individual entries of resolvents and for spectral expansion of unity operator.

In the last part of the book, more complicated matrices with the block structure are considered. Each block is of size $q \times q$, and the number n of blocks increases to infinity. It is assumed that block variables are asymptotically independent, that is, an appropriate generalized mixing condition holds, and a version of the Lindeberg condition is satisfied. For this case, some stochastic canonical equations are obtained that describe, in particular, generalized Wigner multi-mode densities and present a new kind of "SS-laws".

For each of these cases the author presents an extended investiation, in which a number of important properties of spectral functions are studied. Strong law of convergence of $\mu_n(x)$ is proved for a number cases, accompanying equations are analyzed, an invariance principle is offered stating the insensivity of resolvents of large random matrices to distributions. The theorems are proved stating the convergence of quadratic forms of spectra-dependent functionals to constants ("self-averaging").

Appearance of the reviewed monograph undoubtedly presents a remarkable event in the development of theory of random matrices. It summarizes a great deal of different investigations in finding new CSE. The author's results can be of a considerable interest in a number of applications: first, for physicists working in a new branch of physics called *random matrix physics*that is presented today by over than 1000 publications (see [1], for example). Also this book is of an undoubtful interest for a large community of mathematicians, for statisticians working with large covariance matrices, and for specialists in the neuron nets theory.

Apart from results, the investigators working in this region and in its neighborhoods will be much interested in the rich mathematical technique developed by V.Girko and well demonstrated in the monograph. The variety of mathematical discoveries and refined methods developed by V.Girko can consist a good mathematical luggage for young mathematicians sufficient to achive the success.

References

1. T.Guhr, A.Muller-Groeling, and H.Weidenmuller, Random matrix theories in quantum physics: common concepts. Physics Rep., 1998, vol.299, 189–425.